SOME GROUP PRESENTATIONS WITH FEW DEFINING RELATIONS

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ABSTRACT

We consider two classes of groups with two generators and three relations . One class has a similar presentation to groups considered in the paper by C.M.Campbell and R.M.Thomas , *On (2,n)-Groups related to Fibonacci Groups* ,Israel J.Math.,**58** with one generator of order three instead of order two . We attempt to find the order of these groups and in one case find groups which have the alternating group A_5 as a subgroup of index equal to the order of the second generator of the group . Questions remain as to the order of some of the other groups .

The second class has already been considered in the paper `Some families of finite groups having two generators and two relations` by C.M.Campbell , H.S.M.Coxeter and E.F.Robertson (Proc.R.Soc.London A.**357**,423-438(1977)) in which a formula for the orders of these groups was found . We attempt to find simpler formulae based on recurrence relations for subclasses and write Maple programs to enable us to do this . We also find a formula , again based on recurrence relations , for an upper bound for the orders of the groups .

DECLARATIONS

(a) I hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in partial or complete fulfilment of any other degree or professional qualification.

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CERTIFICATION

I hereby certify that the candidate has fulfilled the conditions of the Resolutions and Regulations appropriate to the degree of M.Sc.

Signature of Supervisor Date 30 January 1990

Preface

I would like to thank Dr. C. M. Campbell , under whose supervision this work has been carried out , for his constant encouragement , invaluable guidance and informative lectures on Group Theory .

I am grateful to Dr. E. F. Robertson for his invaluable lectures on Maple which I attended during my first term at St. Andrews .

This thesis is dedicated to my parents whose encouragement and support enabled me to take this degree .

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Chapter 1 : Introduction

This thesis considers two classes of groups with two generators and three relations. The class considered in Chapter 2 has a similar presentation to groups considered in [4], with one generator of order three instead of order two. We attempt to find the order of these groups and in one case find groups which have the alternating group $\mathcal{A}_5$ as a subgroup of index equal to the order of one of the generators of the group.

The class considered in Chapter 3 has already been considered in [2] in which a formula for the orders of these groups was found. We attempt to find formulae involving recurrence relations for the orders of groups in subclasses and write Maple programs to enable us to do this.

Maple

Maple [9] is a symbolic computation package that was designed in 1980 at the University of Waterloo, Canada. It is a sophisticated mathematical calculator but it is also a high level programming language. It is modular - when a task is to be performed then only those parts necessary are called and so it is able to be used on Macintosh Plus micros which have just 1 megabyte of memory. It is thus very useful for undergraduate teaching. There are other symbolic computation systems which have been designed to solve the very large computations appearing in research into relativity, but these require powerful machines.

Maple has a number of packages each of which contain various functions relating to a particular application. These packages have to be called when necessary. The package used in Chapter 3 is the linear algebra package.

If it is possible to find a function that gives the orders of a certain class of groups, then Maple can be used to find relationships between the orders of groups within this class. This is the strategy used in Chapter 3.

Descriptions will now be given of the two processes used for finding presentations of subgroups of finite index that are used extensively in Chapters 2 and 3. For justification of these processes see [6], [7].

The Reidemeister-Schreier algorithm

This is a method of finding a presentation of a subgroup of finite index in a group. For the process to be carried out it is necessary to know a Schreier transversal of the subgroup with respect to a generating set of the group.

Lexicographic ordering

Lexicographic ordering is a method of ordering all the elements of a free group F(X) on a generating set X. It is defined by ordering all the members of the set of generators and their inverses. A reduced word in a free group is a word in X and the inverses of X with the property that no consecutive letters in the word are inverses of each other. (i.e. if $w=x_1 \ldots x_n$ with $x_i \in X \cup X^{-1}$ then $x_i x_{i+1} \neq e$). These reduced words are the elements of the free group. The length of a word in these generators and their inverses is defined to be the number of generators and their inverses appearing in the expression of the word so $abaaba^{-1}bb$ has a appearing 3 times, b also 4 times and a^{-1} once giving a length of 8. If two elements have different lengths then the one with smaller length is said to be the smaller of the two words according to the ordering. For two different elements of the same length the ordering is determined by comparing elements in the same position. The ordering of the two words is defined to be the same as the ordering of the first pair of different elements starting from the left : so if we defined $a < b$ then $abaaba^{-1}bb < abbab^{-1}b^{-1}a^{-1}a^{-1}$.

A Schreier transversal

A Schreier transversal of a subgroup H of F(X) consists of the least elements of each coset of H. It is possible to modify the construction of the Schreier transversal by restricting our attention to just those words with all the terms in the product being group generators rather than both group generators and their inverses. If such words are called positive then the Schreier representative in a coset is the least positive word in that coset. Each coset must contain a positive word providing the rank of the group is finite [7, p.17].

The algorithm

Let us say that we wish to find a presentation of a subgroup H in a group $\langle X \mid R \rangle$. Let $N = \underline{R}$ (the normal closure of the subgroup generated by the elements of R). We find a Schreier Transversal of N in the free group F(X). It is sometimes useful to use coset enumeration to do this (see below). Let the transversal be the set U. The set $UX = \{ux \mid u \in U, x \in X\}$ is formed. Then for each element ux of this set its coset representative $\underline{ux}$ is found, and the set $UX\underline{UX}^{-1} =$

$\{ux\underline{ux}^{-1} \mid u \in U,\ x \in X,\ \underline{ux} \in U,\ ux\underline{ux}^{-1} \in H\}$ is also formed. This set is a generating set of the subgroup H. To find the corresponding relators we first produce the set $URU^{-1}=\{uru^{-1} \mid u \in U,\ r \in R\}$ and then we rewrite the members of this set in terms of the subgroup generators we have found. Thus we have a presentation for the subgroup H.

Useful Result

A useful result for doing the Reidermeister-Schreier process is that if an element in the Schreier transversal has $x_1 \ldots x_{r-1}x_r$ as the least word equal to it then $x_1 \ldots x_{r-1}$ is also in the transversal.

Example

We will use the algorithm to find a presentation of the subgroup $H=\langle a,b^2\rangle$ in $\langle a,b \mid a^3=1=b^6,\ ab=ba\rangle$. Here $X=\{a,b\}$, H is normal as the group is abelian and a transversal is $U=\{1,b\}$.

$UX=\{a,b,ba,b^2\}$ $\therefore$ $UX\underline{UX}^{-1}=\{a,1,bab^{-1},b^2\}$.

Call these generators w,x,y,z.

$URU^{-1}=\{a^3,\ b^6,\ aba^{-1}b^{-1},\ ba^3b^{-1},\ b^6,\ baba^{-1}b^{-2}\} = \{w^3,\ z^3,\ wy^{-1},\ y^3,\ yzw^{-1}z^{-1}\}$. Hence the subgroup has presentation : $\langle y,z \mid y^3=1=z^3, yz=zy\rangle$.

The coset enumeration process

This is a process for finding the index of a finitely generated subgroup of a finitely presented group. It is justified in [1] and [4]. It is not strictly an algorithm as the number of steps required for the process to complete can not be predetermined. However if the subgroup is of finite index in the group then the process must terminate at some stage. So if completion is not attained even after many steps (as is often the case using the computer implementation of the process) then it is still never possible to state that the subgroup must be of infinite index.

Unlike the Reidemeister-Schreier process it is necessary for a generating set of the subgroup to be known. Firstly the words in terms of the group generators are written down without using the common convention of writing the product of an element with itself many times over as that element raised to an integer power. Then the group relators are written in the same form. The number 1 is taken to represent the subgroup and for each of these words, on the row below them, 1 is written just to the left of the word and also just to the right of the word.

Choosing an example as to illustrate the coset enumeration process, at this stage we have :

Generators of H. Relators of G.

	a				a	a	a	a			b	b	b			a	b	a^{-1}	a^{-1}	a^{-1}	b^{-1}
1		1		1					1	1				1	1						1

Now we look for information from these tables. The tables are used to represent information about the cosets. Whenever two cosets i and j are in adjacent columns with x the generator or inverse-generator between the two columns then $i\ x = j$. If we did not use the fact that $i\ x = j$ to place i next to j then this may be new information in which case we would remember it and then could apply it elsewhere in the tables. In this particular example $1\ a = 1$ and so this information can be stored and used to fill spaces in the tables. For example, looking at the first relator of G we can fill in the second column with 1, then the third and then the fourth. We would then have met up with the 1 placed to the right and the potential new information : $1\ a = 1$ is of course already known. We would also use this information to fill the second column of the first row in the third relator of G.

At this stage it is not possible to fill any more spaces, yet spaces remain, so it is now necessary to define another coset. Whenever we have n cosets then the (n+1) st is defined as the product of one of the n with, on the right, a group generator or the inverse of such (so all the cosets are right cosets). i.e. n+1 = j x where $j \in \{1,...,n\}$ and x or x^{-1} is one of the generators of G. Whenever a new coset is defined a new row appears below each group relator, with first and last entries being filled by the number of the new coset (note that new rows do not appear in the subgroup generator tables). Once another coset has been defined, it is possible to fill more spaces and some new relations may be found ; once no more spaces can be filled yet another coset is defined and the process is continued. If at any stage a deduction is made that contradicts a previously known relation : for example if it is known that i x = j but the new deduction is i x = k, then we see from this that the cosets represented by j and k are the same. For convenience we choose the least of j and k (assume this is j here) to name this coset and wherever k appears we now replace it by j. This will often yield even more coincidences : for example if we knew that j y = h and k y = g where y is a group generator then identifying j and k tells us that h=g. In this way, new cosets are defined until all the tables have no spaces. Then we have all the cosets of the subgroup and hence the index of this subgroup. The TC program applies a version of this process to find the index of a subgroup. Coincidence processing is the subject of much study [8, p.14]

In the example we could now define 1 b = 2 to produce :

a	aaaa	bbb	$a\ b\ a^{-1}\ a^{-1}\ a^{-1}\ b^{-1}$
1 1	1 1 1 1 1	1 2 1	1 1 2 2 1
	2 2	2 1 2	2 2

Then 2 b = 3. Applying this to the second relator table we would gain the new information that 3 b = 1 and applying the latter and 1 a = 1 to the third row of the third relator table we would see that 3 a = 3. The second row of the third relator table would tell us that 2 a = 2 and we would have completion :

a	aaaa	bbb	$a\ b\ a^{-1}\ a^{-1}\ a^{-1}\ b^{-1}$
1 1	1 1 1 1 1	1 2 3 1	1 1 2 2 2 2 1
	2 2 2 2 2	2 3 1 2	2 2 3 3 3 3 2
	3 3 3 3 3	3 1 2 3	3 3 1 1 1 1 3

Other applications

Coset enumeration can often be used to help in the Reidemeister-Schreier process by finding a transversal [7, Chptr.4]. After enumeration is completed, we use the coset definitions to find a Schreier transversal. Obviously the representative of the coset 1 is the element 1. We then scan the list of definitions for those cosets defined as the coset 1 multiplied by a group generator and represent these cosets by their corresponding group generator. Next we represent the cosets defined as a group generator multiplied by one of the cosets already considered by the representative of the defining coset multiplied by the defining generator. Continuing in this way we produce a Schreier transversal of the subgroup. In the above example we would thus obtain $\{1,b,b^2\}$.

Coset enumeration can also determine whether a subgroup is normal. It is easy to see that a subgroup is normal iff i h = i for all cosets i and all subgroup generators h (each coset is of the form Hx where H is the subgroup. If H is normal and h is a generator of H then for any x, xhx^{-1} is in H i.e. xh and x are in the same coset i.e. Hxh=Hx. Conversely if g is in the group then if i h = i for all h and i, take i = Hg then ghg^{-1} is in H for all h). Clearly h can be written in terms of the group generators and then i h can be evaluated using the table of the products of the cosets with the group generators.

The modified coset enumeration process

Also used is the modified coset enumeration process ([1], [4]) which not only finds the index of a subgroup, but also a presentation for it. It is modified in the following ways : each of the subgroup generators is given a name, say x_1 to x_r. Here we will deal with coset representatives rather than cosets. So 1 is taken to be the representative of the 1st coset : the identity. The process is continued much as before, except for when a new row is completed. If this new row is in a table headed by a subgroup generator x_i then the equation (*) 1 $w(x_i) = x_i$ 1 is formed where $w(x_i)$ is the generator x_i expressed as a word in the group generators. Let us call the group generators y_1 to y_s and let $w(x_i)= y_{z(1)}^{\pm 1} y_{z(2)}^{\pm 1} \ldots y_{z(u)}^{\pm 1}$. Now we proceed to apply the coset representative information we have to this row. If we know what 1 $y_{z(1)}^{\pm 1}$ is in terms of the other coset representatives : let us say that 1 $y_{z(1)}^{\pm 1}$ = 3 for example, then we replace 1 $y_{z(1)}^{\pm 1}$ by 3 in equation (*) which would become

3 $y_{z(2)}^{\pm 1} \ldots y_{z(u)}^{\pm 1} = x_i$ 1. We would see whether we knew 3 $y_{z(2)}^{\pm 1}$: say that this is d 2 where d is some word in the subgroup generators, so (*) would be 2 $y_{z(3)}^{\pm 1} \ldots y_{z(u)}^{\pm 1} = d^{-1} x_i$ 1. We continue

like this as far as possible : let us say that we obtain 6 $y_{z(4)}^{\pm1}...y_{z(u)}^{\pm1} = e^{-1}x_i$ 1. Then we write this equation as 6 $= e^{-1}x_i$ 1 $y_{z(u)}^{-\pm1}...y_{z(4)}^{-\pm1}$ and rewrite 1 $y_{z(u)}^{-\pm1}$ if possible, continuing with this again as far as possible. Eventually we will obtain the new result, which will be entered in our table of results and used itself : it will be of the form i $y_{z(d)}^{\pm1} = w$ j, where w is a word in x_1 to x_r.

If the completed row is in a table headed by a group relator then the same process is carried out, the only difference being that initially the (*) equation is of the form i w(R) = i where the row starts and finishes with i and w(R) is the word at the top of the table.

If coincidence occurs then as before we identify the coset representatives at all positions. This may give us some more information : for example if we have that

3 $y_5^{-1} = x_{i(1)}{}^{\pm1}x_{i(2)}{}^{\pm1}...x_{i(u)}{}^{\pm1}$ 5 and 6 $y_5^{-1} = x_{j(1)}{}^{\pm1}x_{j(2)}{}^{\pm1}...x_{j(v)}{}^{\pm1}$ 5 then 6= $x_{j(1)}{}^{\pm1}...x_{j(v)}{}^{\pm1}\, x_{i(u)}{}^{-\pm1}...x_{i(1)}{}^{-\pm1}$ 3. Call this 6 = w 3. This is then used to replace the coset representative 6 by 3, we may have for example that 6 $y_5 = w_2$ 4 and 3 $y_5 = w_3$ 4, which would give the subgroup relator w w_3=w_2.

Then when the tables are all completed, all the rows that did not supply any new relations are considered. Each row is of the form i w(R) = i where w(R), the relation at the top of the table is written in terms of the group generators. Then the definitions and deductions are applied to i w(R) = i to produce the relations of the subgroup. An example is given below :-

We will find a presentation of $\langle a,b^2\rangle$ in $\langle a,b|a^3=1=b^6,ab=ba\rangle$.

Subgroup generators Group relators

x	y			
a	b b	a a a	b b b b b b	a b a^{-1} b^{-1}
$\underline{1\ 1}$	1 1	1 1	1 1	1 1

This gives the information 1 a = x 1. Applying this we obtain :

a	b b	a a a	b b b b b b	a b a^{-1} b^{-1}
1 1	1 1	1 1 1 1	1 1	1 1 1

The third table gives no new information. Then we could define coset 2 as 1 b = 2, giving :

a	b b	a a a	b b b b b b	a b a^{-1} b^{-1}
$\underline{1\ 1}$	1 $\underline{2\ 1}$	1 1 1 1	1 2 1	1 1 $\underline{2\ \ 2}$ 1
		2 2	2 1	2 2

The second table tells us that 1 b b = y 1, i.e. 2 b = y 1. The fifth table tells us that 1 $aba^{-1}b^{-1}$ = 1 i.e. x1 $ba^{-1}b^{-1}$ = 1 i.e. x2 $a^{-1}b^{-1}$ = 1. This is x 2 = 1 b a; x 2=2 a which is new information. We can now use these 2 deductions to complete the tables, without collapse in this case, as each of the remaining rows when completed yields no new information.

a	b b	a a a	b b b b b b	a b a^{-1} b^{-1}
$\underline{1\ 1}$	1 $\underline{2\ 1}$	1 1 1 1	1 2 1 2 1 2 1	1 1 $\underline{2\ \ 2}$ 1
		2 2 2 2	2 1 2 1 2 1 2	2 2 1 1 2

Next we examine the rows which have not given any new information (i.e. those rows without an underlined part). The two rows under the a^3 relation : 1 a a a =1 is x1 a a = 1 or x^2 1 a =1 or x^3 1 =1 giving x^3=1, and the other row gives the same subgroup relation. The second row of the fifth table is 2 a b a^{-1} b^{-1} = 2 which is 2 a b = 2 b a : x 2 b = 2 b a : x y 1 = 2 b a : x y 1 = y 1 a : x y 1 = y x 1 or xy=yx. The remaining two rows in the fourth table give the subgroup relation $y^3=1$ and so the subgroup is seen to have index 2 in the group and presentation : $\langle x,y \mid xy=yx,\ x^3=1,\ y^3=1\rangle$.

Situations for usage

The best of the above two methods to use depends on the particular problem. If the coset enumeration involves a lot of coincidences then it is obviously awkward to carry out the modified process. Similarly there is a problem if there are many cosets. Often it isn't known which is the most productive method until the presentations have been found.

The Reidemeister-Schreier method always produces $(|U|-1) \times |X|$ generators. Of course it is often easy to remove many of these. Nevertheless this process produces more generators for the subgroup than existed for the group, and often more than the other process. So attempting to identify a group by finding a presentation of a subgroup, then a presentation of a subgroup of the subgroup and then repeating this process can often soon produce presentations with many generators. In essence the Reidemeister-Schreier process and the modified coset enumeration process are the same method applied in two different ways.

Chapter 2 : Certain (3,n)-Groups

(a) Introduction

In [2] and [4] a class of groups on two generators was considered. The class was those groups with one generator of order 2 and the other of order n and one further defining relation of a special type. This third relation was of (a,b)-length (define this to be the sum of the numbers of powers of a and b occuring in the word) at most 8 and in it the sum of the exponents of the element of order n was zero. Here certain groups of the associated class with generators of order 3 and n and an extra relation of the above type will be considered.

Define $\left({}_{q}^{h}{}_{r}^{i}{}_{s}^{j}{}_{t}^{k}\right)(n) = \langle a,b \mid a^3=1,\ b^n=1,\ a^q b^h a^r b^i a^s b^j a^t b^k = 1\rangle$. Note that as in [2] and [4] we will take $h+i+j+k=0$. Clearly we need only to consider cases when n is positive as if n is zero then add the relation $a=1$ to obtain a free group. Three classes will be considered $\left({}_{q}^{1}{}_{r}^{2}{}_{s}^{-2}{}_{t}^{-1}\right)$, $\left({}_{q}^{1}{}_{r}^{1}{}_{s}^{-1}{}_{t}^{-1}\right)$ and $\left({}_{q}^{2}{}_{r}^{2}{}_{s}^{-2}{}_{t}^{-2}\right)$. In each class only those cases with a degree of symmetry in the sense that for all n the group $\left({}_{q}^{h}{}_{r}^{i}{}_{s}^{j}{}_{t}^{k}\right)(n)$ is isomorphic to $\left({}_{t}^{h}{}_{s}^{i}{}_{r}^{j}{}_{q}^{k}\right)(n)$ will be considered. This condition is introduced because the other groups do not appear to be interesting.

(b) The groups $\left({}_q{}^1{}_r{}^2{}_s{}^{-2}{}_t{}^{-1}\right)$.

Let $q=e_1$, $r=e_2$, $s=e_3$ and $t=e_4$. Let the group $\left({}_q{}^1{}_r{}^2{}_s{}^{-2}{}_t{}^{-1}\right)$ be (e_1,e_2,e_3,e_4). Note that (e_1,e_2,e_3,e_4) is isomorphic to $(-e_1,-e_2,-e_3,-e_4)$ and also to $(-e_1,-e_4,-e_3,-e_2)$. So when all the e_i are non-zero, the cases to consider are $(1,1,1,1)$, $(1,-1,1,-1)$ and $(1,1,-1,-1)$.

Note the reasons for the above statements :- The case $(1,1,1,-1)$ is not isomorphic to $(-1,1,1,1)$ as when $n = 2$ the former is $\langle a,b \mid a^3=1,\ b^2=1,\ aba=b\rangle$ which is of order 6 whereas the latter is C_2. The fact that the class $(1,1,-1,1)$ is not in general isomorphic to the class $(1,-1,1,1)$ is seen by considering the $n=4$ groups. Use the coset enumeration algorithm to find the index and presentation of the subgroup $\langle a,\ bab^{-1},\dots,\ b^3ab^{-3}\rangle$ in $(1,1,-1,1)$. This is therefore seen to be $\langle x_1,\dots,x_4 \mid x_i^3=1,\ x_{i+2}x_{i+1}x_{i+2}=x_{i+4}\rangle$ and the Todd-Coxeter coset enumeration program then tells us that the element x_1 has index 25 and so the group is finite. However carrying out the same process on the group $(1,-1,1,1)$ we obtain the subgroup

$\langle x_1,\dots,x_4 \mid x_i^3=1,\ x_{i+4}x_{i+2}x_{i+1}=x_{i+2}\rangle$. It is seen that $x_1=x_2^{-1}x_4^{-1}x_2$ and $x_3=x_4^{-1}x_2^{-1}x_4$ so use these to remove the generators x_1 and x_3. Letting $u=x_2$ and $v=x_4$ we obtain the presentation

$\langle u,v \mid u^3=1,\ v^3=1,\ u^{-1}v^{-1}uv^{-1}u^{-1}vuv^{-1}uv=1\ \rangle$. Note that the last two relations that were obtained were the same so one has been removed. Now note that in the abelianised group $u=v$ and so this group is C_3 so this enables us to write a transversal for the derived group $\{1,u,u^2\}$. Now carrying out the Reidemeister-Schreier process the presentation of the derived group, with $a=vu^{-1}$, $b=uvu^{-2}$ and $c=u^2v$, is $\langle a,b,c \mid abc=1,\ b^{-2}ab^{-1}c=1,\ c^{-2}bc^{-1}a=1,\ a^{-2}ca^{-1}b=1\rangle$. Removing c and a redundant relation we obtain $\langle a,b \mid a=b^2ab^2,\ b=a^2ba^2\rangle$. Add $b^2=1$ and $a^2=1$ to obtain $\langle a,b \mid b^2=1,\ a^2=1\rangle$. So it is seen that this group has infinite order.

No cases exist with just one of the e_i of value zero. If two are zero then the cases are those of the form $(0,e_2,e_3,0)$ and $(e_1,0,0,e_4)$. When three of the e_i are zero, the group is C_n and when all 4 are zero the group is the free product of C_3 and C_n and is hence infinite.

(6.1) The case (e,0,0,f).

These are the groups $\langle a,b \mid a^3=1,\ b^n=1,\ a^eba^f=b\rangle$. If $e=-f$ then they are $C_3\times C_n$. If $e=f$ then when n is odd $a^{-1}ba^{-1}=b$ and $aba=b$ so $a^{-1}b^2a=b^2$ and hence $b^n=aba(a^{-1}b^2a)^{(n-1)/2}=ab^na$ which with $a^3=1$ implies that the group is C_n. When n is even carry out the modified coset enumeration algorithm on $\langle b\rangle$ to show that $\langle b\rangle$ has index 3 and is C_n. Define 1 a = 2 and 2 a = 3.

b	a b a b^{-1}	a a a
1 1	1 2 3 1 1	1 2 3 1
	2 3 2 3 2	
	3 1 1 2 3	

Let the subgroup generator b be x then 1 x = 1, 2 x = 3 and 3 x = 2 and so when n is even the group is of order 3n.

(6.2) The case (1,1,-1,-1).

The group obtained when $e_1,e_2=1$ and $e_3,e_4=2$ is

$\langle a,b \mid a^3=1,\ b^n=1,\ abab^2a^{-1}b^{-2}a^{-1}b^{-1}=1\rangle$. Call this group G_n .The orders of these groups were first of all investigated using the Todd-Coxeter program on the Vax System of the University of St Andrews Computing Laboratory with the maximum number of cosets being 30,000 and the index of the subgroup $\langle b\rangle$ being found. For the values of n from 1 to 5 the following results were obtained :-

n	index of $\langle b\rangle$	max. number of cosets defined
1	3	9
2	3	5
3	27	32
4	39	63
5	93	1342

However for values of n above 5 the coset enumeration was not completed. Several of these cases were considered with a considerably higher maximum number of cosets (150,000) but again, no results were obtained. This and the contrasting ease with which the results for n from 1 to 5 were obtained suggested that these groups are infinite.

An attempt was made to prove that G_6 was infinite :-

Lemma 1

The group G_n has $H_n=\langle x,y,z \mid x^n=y^n=z^n=1, xy^2=yz^2=zx^2\rangle$ as a subgroup of index 3.

Proof:-

The modified Todd-Coxeter algorithm is used with the subgroup being $\langle b,aba^{-1},a^2ba^{-2}\rangle$.

Let $x=b, y= aba^{-1}, z= a^2ba^{-2}$. Define 1 a= 2, 2 a= 3. Then the subgroup generators give :-

b	a b a^{-1}	a a b $a^{-1}a^{-1}$
$\underline{1\ 1}$	1 $\underline{2\ 2}$ 1	1 2 $\underline{3\ 3}$ 2 1
1b=x1	2b=y2	3b=z3

Also the relation $a^3=1$ gives that $3a=1$. So completion occurs and the other two generators give the five relations.

Lemma 2

The group H_n has $B_n=\langle b_i \mid b_i^n=1,\ b_{i+1}b_{i+2}=b_ib_{i+2}b_{i+3}{}^2b_{i+4},\ i=0,1,\ldots,n\rangle$ as a subgroup of index n.

Proof:- Step 1:-

$B_n=\langle xy^{-1}, x^{-1}y, yz^{-1}, y^{-1}z\rangle$ is a normal subgroup of H_n.

Note that $z^{-1}x,\ zx^{-1}\in B_n$.

$x(xy^{-1})x^{-1}=x^2y^{-1}x^{-1}=z^{-1}xy^2y^{-1}x^{-1}\in B_n$, $x^{-1}(xy^{-1})x\in B_n$, $z(xy^{-1})z^{-1}=zx^{-1}x^2y^{-1}z^{-1}=zx^{-1}zx^{-1}y^2y^{-1}z^{-1}\in B_n$, $z^{-1}(xy^{-1})z\in B_n$, $y(xy^{-1})y^{-1}=yx(x^{-2}z^{-1}x)\in B_n$, $y^{-1}(xy^{-1})y\in B_n$. Therefore, by symmetry, all other conjugates of B_n are in H_n.

Step 2:-

The Reidemeister-Schreier process is now used to find a presentation of B_n.

Clearly, as B_n is normal in H_n, the presentation of the factor group is $\langle x \mid x^n=1\rangle$ and a transversal of B_n is $U=\{1,x,x^2,\ldots,x^{n-1}\}$. Take X to be the set of generators of H_n. Then the set obtained when each element of UX is multiplied by the inverse of its coset representative is $\{x^n,\ x^iyx^{-i-1},\ x^izx^{-i-1}\}$. Call these elements α, β_i, γ_i

respectively. Then the set URU^{-1} in terms of these generators is obtained and on eliminating α, β_i the required presentation is found.

Now the task was to prove that B_6 is infinite. To attempt this the homomorphic image of B_6 obtained by setting $\beta_i=\beta_{i+3}$ was considered. This is $\langle a,b,c \mid a^6=b^6=c^6=1,\ ab=cbc^2a,\ bc=aca^2b,\ ca=bab^2c \rangle$ and appears to be infinite. However, using the Sun 3/260 with a maximum of 800,000 cosets, Dr. Campbell found that this was of order 27. (This is not easy to prove directly. To attempt to factor by one generator is of no use, so use $N(\langle a,b\rangle)$. Clearly $c^3=(cbc^{-1})^{-1}(aba^{-1}) \in N(\langle a,b\rangle)$, but as $G/N(\langle a,b\rangle) = C_3$, clearly $c \notin N(\langle a,b\rangle)$. Therefore $\{1,c,c^2\}$ is a tranversal for $N(\langle a,b\rangle)$ and applying Reidemeister-Schreier a presentation for this group is found:-

$\langle a_0,a_1,a_2,b_0,b_1,b_2,c \mid a_i^6,\ b_i^6,\ c^2,\ a_0b_0=b_1ca_0,\ a_1b_1=b_2ca_1,\ a_2b_2=cb_0a_2,\ b_2c=a_2ca_0^2b_0,\ b_0=a_0a_1^2b_1,\ b_1=a_1a_2^2b_2,\ ca_0=b_2a_2b_2^2c,\ a_1=b_0a_0b_0^2,\ a_2=b_1a_1b_1^2\rangle$. But proceeding from this leaves too many generators or too long relations.)

An alternative method of looking at the group G_n was then adopted.

__Lemma 3__

The group G_n has $X_n=\langle x_1,\dots,x_n \mid x_i^3=1,\ x_ix_{i+1}=x_{i+1}x_{i+3}\rangle$ as a subgroup of index n.

Proof:- Use the Reidemeister-Schreier algorithm and the normal subgroup $\langle a, bab^{-1},\dots, b^{n-1}ab^{1-n}\rangle$. Clearly a transversal is $\{1,b,\dots,b^{n-1}\}$. The set $UX(\mathbf{UX})^{-1}$ is $\{a, bab^{-1},\dots, b^{n-1}ab^{1-n}, b^n\}$ with URU^{-1} giving the required relations.

Corollary 3.1

X_1 is C_3.

Corollary 3.2

X_2 is C_3.

Corollary 3.3

X_3 is $C_3 \times C_3 \times C_3$.

Corollary 3.4

X_4 is of order 39.

Proof :-

A presentation for X_4 is

$\langle x_1, x_2, x_3, x_4 \mid x_i^3=1,\ x_1x_2=x_2x_4,\ x_2x_3=x_3x_1,\ x_3x_4=x_4x_2,\ x_4x_1=x_1x_3 \rangle$.

Note that $\langle x_2x_4^{-1}, x_4^{-1}x_2 \rangle$ is a normal subgroup of X_4 of index 3. (This can either be seen by listing conjugates or by carrying out coset enumeration and examining the coset table.) Clearly a transversal of the subgroup is $\{ 1, x_2, x_2^2 \}$ as x_2 being in the subgroup would mean that the subgroup was the whole group. Now apply Reidemeister-Schreier to obtain $\langle a,b,c \mid abc=1,\ bc^{-2}b=a,\ ac^2=b^2,\ ca^{-2}c=b,\ ba^2=c^2,\ ab^{-2}a=c,\ cb^2=a^2 \rangle$. Eliminate a using $a=c^{-1}b^{-1}$. Then $b^2cb=c^2$ and $bcb^2=c^2$ give that $bc=cb$, and the group is seen to be C_{13}.

Note 3.5

X_5 is of order 93.

A presentation for X_5 is $\langle x_1, x_2, x_3, x_4, x_5 \mid x_i^3=1,\ x_1x_2=x_2x_4,\ x_2x_3=x_3x_5,\ x_3x_4=x_4x_1,\ x_4x_5=x_5x_2,\ x_5x_1=x_1x_3 \rangle$. Note that $\langle x_2x_4^{-1}, x_4^{-1}x_2 \rangle$ is a normal subgroup of X_5 of index 3 (most easily seen using coset enumeration). Again $\{ 1, x_2, x_2^2 \}$ is a transversal of the subgroup.

Apply Reidemeister-Schreier to get a presentation of this subgroup $\langle a,b \mid a^3b^3a^2b=1,\ a^2=bab^4abab \rangle$ but it is then not easy to proceed from here. Alternatively it is very easy to find a 2-generator 4-relation presentation using the modified coset enumeration algorithm but it is then difficult to proceed further.

This method doesn't give anything for X_6.

The following was then considered :-

Consider $H_n = \langle x,y,z \mid x^n=y^n=z^n=1,\ xy^2=yz^2=zx^2\rangle$. Eliminate x to get $\langle y,z \mid (z^2y^{-1})^n=1,\ y^n=1,\ z^n=1,\ yz^2=zyz^2y^{-1}z^2y^{-2}\rangle$. The last relation is $z^{-1}(yz)z=y(z^2y^{-1})^2y^{-1}$. Therefore if n is even $(yz)^{n/2}=1 \Leftrightarrow (z^2y^{-1})^n=1$ and $H_n = \langle y,z \mid (yz)^{n/2}=1,\ y^n=1,\ z^n=1,\ yz^2=zyz^2y^{-1}z^2y^{-2}\rangle$. So using $zyz=y^{-1}z^{-1}y^{-1}$ we get $H_6 = \langle y, z \mid (yz)^3=1,\ y^6=1,\ z^6=1,\ yzy^2z^2y^2z^{-2}yz^{-1}\rangle$. (Note that G_6 is $\langle a,b \mid a^3=1,\ b^6=1,\ abab^2a^2b^{-2}a^2b^{-1}=1\rangle$ and this is isomorphic to $\langle a,b \mid a^3=1,\ b^6=1,\ aba^2b^2a^2b^{-2}ab^{-1}=1\rangle$ under the isomorphism given by taking $a \to a^{-1}$ and $b \to b$.)

Now carry out coset enumeration on H_6.

Lemma 4

H_6 has $A_6 = \langle a_0,a_1,a_2 \mid a_0^6=a_1^6=a_2^6=1,\ (a_0a_1a_2)^2=1,\ a_0^2a_1a_2a_1a_0a_1^2 = a_1a_2,\ a_1^2a_0a_1a_2^2a_1a_2=a_0a_1,\ a_0a_2^2a_0^2a_1a_2=a_1\rangle$ as a subgroup of index 3.

Proof :-

Carry out the modified coset enumeration algorithm with the subgroup $\langle y,\ zyz^{-1},\ z^2yz^{-2}\rangle$. Let $a_i = z^iyz^{-i}$ for i from 0 to 5. Define $1\ z = 2$ and $2\ z = 3$. The generator a_i then yields the information that $i+1\ y = a_i\ \ i+1$.

$$\begin{array}{cccccccccl} z & z\ldots & z & y & z^{-1}\ldots & z^{-1} & z^{-1} & & & : a_i \\ 1 & 2 & 3\ldots & \underline{i+1} & \underline{i+1} & \ldots & 3 & 2 & 1 & 1\ z^i y z^{-i} = a_i\ 1 \end{array}$$

i.e. $i+1\ y= a_i\ i+1$.

The relation $(yz)^3=1$ gives that $3\ z= a_2^{-1}a_1^{-1}a_0^{-1}\ 1$. The relation $y^6=1$ gives $a_i^6=1$ and the relation $z^6=1$ gives that $(a_0a_1a_2)^2=1$. The other relations are obtained from $yz^2=zyz^2y^{-1}z^2y^{-2}$.

It is now difficult to proceed from here. The Todd-Coxeter alogorithm with the maximum number of cosets being 300000 again does not produce completion. The homomorphic image obtained by setting $a_0a_1a_2=1$ is however seen to be $C_3 \times C_3$. Call this image ${}^hA_6$.

Corollary 4.1

${}^hA_6$ is $C_3 \times C_3$.

Proof :-

Using Lemma 4 a presentation for ${}^hA_6$ is

$\langle a_0,a_1,a_2 \mid a_0^6=1,\ a_1^6=1,\ a_2^6=1,\ a_0a_1a_2=1,\ a_ia_{i+1}^2a_i^2a_{i+1}=1\rangle$. (Note that $a_{i+3}=a_i$). Now use the 4th relation to eliminate a_0 to get

$\langle a_1,a_2 \mid (a_1a_2)^6=1,\ a_1^6=1,\ a_2^6=1,\ a_1=a_2^2a_1a_2,\ a_1a_2^2a_1^2a_2=1,\ a_2=a_1a_2a_1^2\rangle$. Now multiplying the 4th relation by a_1^2 and applying the last one gives $a_1^3=a_2^3$. So the 4th relation becomes $a_2a_1=a_1^4a_2$ which is $a_2=a_1^2a_2a_1$. Using the last relation we see that the group is abelian and therefore is $C_3 \times C_3$.

In general modified coset enumeration can be carried out on H_n to produce a presentation for A_n. However this does not yield a general presentation. If the relation $a_0a_1\ldots a_{n/2-1}=1$ is added to A_n then the associated groups ${}^hA_n$ are produced and another of these was considered.

Lemma 5

If n is even then H_n has

${}^hA_n = \langle a_0,\ldots a_{n/2-1} \mid a_i^n=1,\ a_0a_1\ldots a_{n/2-1}=1,\ a_ia_{i+1}{}^2a_{i+3}{}^2a_{i+1}=1\rangle$ as a homomorphic image of a subgroup of H_n of index n.

Proof :-

Carry out the modified coset enumeration algorithm with the subgroup $\langle y, zyz^{-1},\ldots, z^{n/2-1}yz^{1-n/2}\rangle$ as in lemma 4. Add the relation $a_0a_1\ldots a_{n/2-1}=1$ to the subgroup relations. Again let $a_i = z^iyz^{-i}$. Define $i\ z= i+1$ for cosets $i =1,..,n$. The new relation ensures that $n\ z = 1$ and the required presentation of ${}^hA_n$ is obtained.

Corollary 5.1

${}^hA_8$ is of order 392.

Proof :- We proceed by using several lemmas.

Lemma 5.1.1

${}^hA_8 = \langle x,z \mid x^8=1,\ x^4=z^4,\ zxz=x^3,\ x=(z^2x^2)^3z(x^2z^2)^3\rangle$.

Proof :-

A presentation for ${}^hA_8$ is

$\langle w,x,y,z \mid w^8=x^8=y^8=z^8=wxyz=wx^2z^2x=xy^2w^2y=yz^2x^2z=zw^2y^2w=1\rangle$.

Eliminate $w=x^{-1}z^{-2}x^{-2}$ and $y=z^{-1}x^{-2}z^{-2}$ to get

$\langle x,z \mid (x^3z^2)^8 = x^8 = (z^3x^2)^8 = z^8 = xz^2xzx^2z =$

$z^2x^2zx^2z^2x^3.z^2xz^2x^2z^3x^2z.x^{-1} = x^2.z^2xz^2x^2z^3x^2z.x^2z^2x^3z^2xz^{-1} =1\rangle$.

Note that the last two relations have been split to show that they have a common string. Comparing these strings gives us that

$z(zx^2zx)(xz^2xz) = x^3z^2x^3z^2x$ (r1).

Now replace the 7th relation by this. Apply the 5th relation to the left hand side of r1 to get

$$zx^{-1}z^{-3}x^{-2} = x^3z^2x^3z^2x \tag{r2}$$

and replace r1 by this. Note that r2 is

$$zx^{-1}z^{-1} = (x^3z^2)^3 \tag{r3}$$

and by symmetry

$$xz^{-1}x^{-1} = (z^3x^2)^3 \tag{r4}.$$

Now r3 gives that $zx^{-3}z^{-1} = (x^3z^2)^9$ i.e. $z = x^3z^3x^3$ using the 1st relation and similarly r4 gives that $x = z^3x^3z^3$. Together these give

$$x^4 = z^4 \tag{r5}.$$

Now $x^4 = z^4$ and $x^8 = 1$ acting on $z = x^3z^3x^3$ imply that

$$zxz = x^3 \tag{r6}.$$

Add r5 and r6 to the relations. Now r3 and r4 (which is obtainable from the 5th, 6th and 7th and the 7th is a consequence of r2, 5th and 6th) together with the 2nd, the 4th, r5 and r6 make the 1st and 3rd ones redundant (obtain $zx^{-3}z^{-1} = (x^3z^2)^9$ from r3 and apply r6 and then r5 and the 2nd to obtain the 1st; apply r6 to r4 to get $x(x^{-3}z) = (z^3x^2)^3$ or $z^4 = (z^3x^2)^4$ and then use the 4th to obtain the third) and r5 and the 2nd make the 4th redundant so what we have is

$\langle x,z \mid zx^{-1}z^{-3}x^{-2} = x^3z^2x^3z^2x,\ x^4 = z^4,\ zxz = x^3,\ x^8 = xz^2xzx^2z = 1,$ $z^2x^2zx^2z^2x^3z^2xz^2x^2z^3x^2zx^{-1} = 1\rangle$.

Now $x^8=1$ gives that $(zxz)(zxz)x^2=1$ which is $xz^2xzx^2z=1$ so remove $xz^2xzx^2z=1$. Multiply r6 by x to get $xzx=z^3$ then apply that and r6 to

$$\begin{aligned}
z^2x^2zx^2z^2x^3z^2xz^2x^2z^3x^2zx^{-1} &= z^2x^2zx^2z^3xz^3xz^2x^3zx^3zx^{-1} \\
&= z^2x^2zx^3zx^3zx^2z^3xz^3xz^2x^{-1} \\
&= z^2x^2z^2xz^3xz^2x^3zx^3zx^2z^2x^{-1} \\
&= z^2x^2z^2x^2zx^2z^3xz^3xz^2x^2z^2x^{-1} \\
&= z^2x^2z^2x^2zx^3zx^3zx^2z^2x^2z^2x^{-1} \\
&= z^2x^2z^2x^2z^2xz^3xz^2x^2z^2x^2z^2x^{-1}
\end{aligned}$$

$= z^2x^2z^2x^2z^2x^2zx^2z^2x^2z^2x^2z^2x^{-1}$

$= 1$

i.e. $x = (z^2x^2)^3z(x^2z^2)^3$.

So replace the last relation by this .Take $x^3z^2x^3z^2x^3z^3x$ and apply the same two relations

$$\begin{aligned} x^3z^2x^3z^2x^3z^3x &= zxz^3x^3z^2x^4zx^2 \\ &= zxz^4xz^3x^4zx^2 \quad (x^4=z^4 \text{ and } x^8=1) \\ &= z \ . \end{aligned}$$

So it is seen that r2 is redundant giving the required presentation.

Lemma 5.1.2

${}^hA_8$ has $B = \langle b,c,d \mid c^4=1,\ d=bdb,\ c=bcb,\ b^7=1,\ d^2=c^2,\ c=d(cd)^6\rangle$ as a subgroup of index 2.

Proof :-

Step 1

Note that the subgroup $B = \langle x^2, z^2, xzx^{-1}z^{-1}\rangle$ has index 2 as is easily seen by coset enumeration. Therefore it is a normal subgroup. Clearly $x \notin B$ as if so then by symmetry $z \in B$ and the index would not be 2. So $\{ 1,x \}$ is a transversal. Now using the Reidemeister-Schreier method the presentation $\langle b,c,d \mid c^4=1,\ c^2=(bd)^2,\ bcb=c,\ c=d(bcd)^3(cbd)^3,\ c^2=dbdb,\ (bdc)^3(bcd)^3b=1,\ d^2=c^2\rangle$ is found.

Step 2

Note that $c^2 = dbdb \Rightarrow d = bdb$ (r1).

Therefore add this relation and remove the 2nd and 5th relations.

Note that $(bdc)(bcd) = bdc^2b^{-1}d = bd^3b^{-1}d = bd^{-1}b^{-1}d = b^2$.

In fact $(bdc)b^n(bcd) =$ (Use $bc=cb^{-1}$.)

$=bdcb^{n+1}cd$ (Use $d^2=c^2$, $c^4=1$.)

$=bdc^2b^{-n-1}d$ (Use $d^{-1}b^{-1}=bd^{-1}$.)

$=bd^{-1}b^{-n-1}d$

$=b^{n+2}.$

Therefore it is seen that $(bdc)^3(bcd)^3b = 1$ is equivalent to $b^7=1$. Now consider $(bcd)(cbd) = bcdb^{-1}cd = bcbdcd = cdcd$. Also $(bcd)cdcd(cbd) = bcbdcdcdcd = cdcdcdcd$. (Use $db^{-1}=bd$ and $cb=b^{-1}c$ which give that $dcb=db^{-1}c=bdc$.) Furthermore $(bcd)cdcdcdcd(cbd) = b(cd)^6b^{-1}$. So it is seen that $c = d(bcd)^3(cbd)^3$ may be replaced by $c = d(cd)^6$ and we have the required presentation.

Lemma 5.1.3

B has $C = \langle x,y \mid x^4=1, x^2=y^2, x=y(xy)^6 \rangle$ as a subgroup of index 7.

Proof :-

Carry out the modified coset enumeration algorithm using the subgroup $\langle c,d \rangle$.

Let $1\ c = x1$ and $1\ d = y1$. Then define $i\ b= i+1$ for i from 1 to 6. The relation $b^7=1$ gives that $7\ b = 1$, $d=bdb$ gives that $i\ d = y(9-i)$ for i from 2 to 6 and $c=bcb$ gives that $i\ c = x(9-i)$ for i from 2 to 6. The other relations then give the required presentation.

Proposition 5.1.4

C is of order 28.

Proof :-

To see this use the modified coset enumeration alogorithim to find a presentation for the subgroup $\langle x \rangle$, defining cosets $1\ y =2$, $2\ x = 3$, $3\ y = 4$, ..., $6\ x = 7$. There are no coincidences. The subgroup is C_4 and has index 7.

Hence ${}^hA_8$ has the required order.

(6.3) The case (1,1,1,1).

These are the groups $G_n = \langle a,b \mid a^3=1,\ b^n=1,\ abab^2ab^{-2}ab^{-1}=1\rangle$. Using the Todd-Coxeter program it can be quickly conjectured that for values of n divisible by 3, G_n has order 60n and otherwise the order is n.

Lemma 6

G_n has the group $X_n = \langle x_1,\ldots, x_n \mid x_i^3=1,\ x_ix_{i+1}x_{i+3}x_{i+1}=1 \rangle$ as a subgroup of index n.

Proof :-

This is easily seen using coset enumeration. Let $X_n =$

$\langle a,\ bab^{-1},\ldots, b^{n-1}ab^{1-n} \rangle$ where the generators are named $x_1,\ldots, x_n$. Define i b = i+1 for i from 1 to n-1. Then the generator x_i gives that i a = x_i i for i from 1 to n. The relation $b^n=1$ implies that n b = 1 and the other relations give the full presentation.

Corollary 6.1

X_1 is trivial.

Corollary 6.2

X_2 is trivial.

Corollary 6.3

X_3 is the alternating group $\mathcal{A}_5$.

Proof :-

X_3 is $\langle x_1,x_2,x_3 \mid x_i^3=1,\ x_3x_1x_3x_1=1,\ x_1x_2x_1x_2=1, x_2x_3x_2x_3=1 \rangle$. Define $s=x_1x_2x_3^{-1}x_2^{-1}x_1^{-1}$ and $t=x_1$ then

$$\begin{aligned} sts^{-1} &= x_1x_2x_3^{-1}x_2^{-1}x_1x_2x_3x_2^{-1}x_1^{-1} \\ &= x_1x_2^{-1}x_3x_1x_3^{-1}x_2x_1^{-1} \\ &= x_1x_2^{-1}x_1^{-1}x_3x_2x_1^{-1} \end{aligned}$$

$= x_1^{-1}x_2x_3x_2x_1^{-1}$

$= x_3.$

Also $sts^{-1}.t^{-1}s^{-1}t = x_3x_2x_3x_2^{-1} = x_2$. So $\langle s,t \rangle = X_3$. Clearly $s^3=1$, $t^3=1$, $(ts^{-1}ts)^2 = s^{-1}(sts^{-1}.t)^2s = s^{-1}(x_3x_1)^2s = 1$. Consider $(st)^5 = (x_1x_2x_3^{-1}x_2^{-1})^5$.

Now $x_1x_2x_3^{-1}x_2^{-1}.\ x_1x_2x_3^{-1}x_2^{-1} = x_1x_2^{-1}x_3x_1x_2x_3^{-1}x_2^{-1}$

$$
\begin{aligned}
&= x_1x_2^{-1}x_3x_2^{-1}x_1^{-1}x_3^{-1}x_2^{-1}\\
&= x_1x_2^{-1}x_3x_2^{-1}x_3x_1x_2^{-1}\\
&= x_1x_2x_3^{-1}x_2x_3x_1x_2^{-1}\\
&= x_1x_2x_3x_2^{-1}x_1x_2^{-1}\\
&= x_1x_3^{-1}x_2x_1x_2^{-1}\\
&= x_1x_3^{-1}x_1^{-1}x_2\\
&= x_1^{-1}x_3x_2\,.
\end{aligned}
$$

So $(x_1x_2x_3^{-1}x_2^{-1})^5 = x_1^{-1}x_3x_2\,.x_1x_2x_3^{-1}x_2^{-1}\,.x_1^{-1}x_3x_2$

$$
\begin{aligned}
&= x_1^{-1}x_3x_1^{-1}x_3^{-1}x_2^{-1}x_1^{-1}x_3x_2\\
&= x_1^{-1}x_3^{-1}x_1x_2^{-1}x_1^{-1}x_3x_2\\
&= x_1^{-1}x_3^{-1}x_1^{-1}x_2x_3x_2 = 1.
\end{aligned}
$$

Therefore X_3 is a homomorphic image of

$\mathcal{A}_5 = \langle s,t \mid s^3=1, t^3=1, (ts^{-1}ts)^2=1, (st)^5=1 \rangle$ [5]. Now in $\mathcal{A}_5$ call $x_1=(123)$, $x_2=(124)$ and $x_3=(125)$. Then $\langle x_1, x_2, x_3 \rangle$ is $\mathcal{A}_5$ and $x_i^3=1$, $x_3x_1x_3x_1=1$, $x_1x_2x_1x_2=1$ and $x_2x_3x_2x_3=1$. So $\mathcal{A}_5$ is also a homomorphic image of X_3 proving the result.

Corollary 6.4

X_4 is trivial.

Proof :-

$X_4 = \langle x_1,x_2,x_3,x_4 \mid x_i^3=1,\ x_1x_2x_4x_2=1,\ x_2x_3x_1x_3=1,\ x_3x_4x_2x_4=1,$
$x_4x_1x_3x_1=1 \rangle$.

Now $x_1x_2x_4x_2x_4 = x_4$, therefore $x_1x_2 = x_4x_3$. Similarly $x_2x_3 = x_1x_4$. $x_1x_2x_3 = x_4x_3^{-1}$ and $x_1^{-1}x_4$ giving that $x_1x_4 = x_4x_3$. So $x_2 = x_4$ and $x_1x_2x_4x_2 = 1$ gives that $x_1 = 1$.

Corollary 6.5

X_5 is trivial.

Proof :-

$X_5 = \langle x_1,x_2,x_3,x_4,x_5 \mid x_i^3=1,\ x_1x_2x_4x_2=1,\ x_2x_3x_5x_3=1,\ x_3x_4x_1x_4=1,$
$x_4x_5x_2x_5=1,\ x_5x_1x_3x_1=1\rangle$.

$1= x_4^3= (x_5^{-1}x_2^{-1}x_5^{-1})^3 = (x_5^{-1}x_3x_5x_3x_5^{-1})^3 = (x_5x_3)^6$

$$\begin{aligned}1 &= x_5x_1x_3x_1\\ &= x_5x_2^{-1}x_4^{-1}x_2^{-1}x_3x_2^{-1}x_4^{-1}x_2^{-1}\\ &= x_5x_3x_5x_3x_4^{-1}x_3x_5^{-1}x_3\, x_4^{-1}x_3x_5x_3\\ &= x_5x_3x_5x_3x_5x_2x_5x_3x_5^{-1}x_3x_5x_2x_5x_3x_5x_3\\ &= x_5x_3x_5x_3x_5x_3^{-1}x_5^{-1}x_3^{-1}x_5x_3x_5^{-1}x_3x_5x_3^{-1}x_5^{-1}x_3^{-1}x_5x_3x_5x_3.\end{aligned}$$

Now apply $(x_5x_3)^6 =1$ to get

$$\begin{aligned}1 &=x_3x_5^{-1}x_3^{-1}x_5x_3x_5^{-1}x_3x_5x_3^{-1}x_5^{-1}x_3x_5^{-1}\\ &=x_3x_5x_3x_5^{-1}x_3x_5x_3^{-1} = x_3. \qquad \text{(apply } (x_5x_3^{-1})^3\text{)}\end{aligned}$$

Theorem 6.6

X_6 is the alternating group $\mathcal{A}_5$.

Proof :-

$X_6 = \langle x_1,x_2,x_3,x_4,x_5,x_6 \mid x_i^3=1,\ x_1x_2x_4x_2=1,\ x_2x_3x_5x_3=1,\ x_3x_4x_6x_4=1,$
$x_4x_5x_1x_5=1,\ x_5x_6x_2x_6=1,\ x_6x_1x_3x_1=1 \rangle$.

$1 = x_1^3$

$= (x_2x_4^{-1})^3$

$=x_3^{-1}x_5^{-1}x_3^{-1}x_4^{-1}.\ x_2x_4^{-1}.\ x_3^{-1}x_5^{-1}x_3^{-1}x_4^{-1}$

$=x_4x_6x_4x_5^{-1}x_4x_6x_2x_6x_4x_5^{-1}x_4x_6$

$=x_4x_6x_4x_5^{-1}x_4x_5^{-1}x_4x_5^{-1}x_4x_6$

$=x_4x_6x_4x_6$ (Now use that $(x_4x_5^{-1})^3=1$) .

Therefore $x_3=x_6$ and $X_6= X_3$.

Corollary 6.7

X_7 is trivial.

Proof :-

$X_7 = \langle x_1,x_2,x_3,x_4,x_5,x_6,x_7 \mid x_i^3=1,\ x_1x_2x_4x_2=1,\ x_2x_3x_5x_3=1,$

$x_3x_4x_6x_4=1,\ x_4x_5x_7x_5=1,\ x_5x_6x_1x_6=1,\ x_6x_7x_2x_7=1,\ x_7x_1x_3x_1=1 \rangle.$

Now $1 = x_6x_7x_2x_7$

$= x_6x_1^{-1}x_3^{-1}x_1^{-1}x_2x_1^{-1}x_3^{-1}x_1^{-1}$

$= x_6x_1^{-1}x_3^{-1}x_2^{-1}x_1x_2^{-1}x_3^{-1}x_1^{-1}$ (Use $(x_1x_2^{-1})^3=1$)

$= x_6x_1^{-1}x_5x_3x_1x_3x_5x_1^{-1}$

$= x_5x_6x_5x_3x_1x_3x_5x_6x_5.$

(Use $x_1^3=1 \Rightarrow (x_6x_5x_6)^3=1 \Rightarrow (x_6^{-1}x_5)^3=1 \Rightarrow (x_5x_6x_5)^3=1.$)

Therefore

$x_3x_1x_3=x_5x_6x_5.$ (1).

This gives $x_6^{-1}x_4^{-1}x_1x_4^{-1}x_6^{-1} = x_4x_5x_6x_5x_4$

$= x_5^{-1}x_7^{-1}x_6x_7^{-1}x_5^{-1}$

$= x_5^{-1}x_6^{-1}x_7x_6^{-1}x_5^{-1}$.

So $x_4^{-1}x_1x_4^{-1} = x_5x_6^{-1}x_5x_6x_7x_6x_5x_6^{-1}x_5$

$= x_5x_6^{-1}x_5x_4x_2x_4x_5x_6^{-1}x_5$ (using (1)).

$$= x_5 x_6^{-1} x_7^{-1} x_5^{-1} x_2 x_5^{-1} x_7^{-1} x_6^{-1} x_5$$

$$= x_5 x_7 x_2 x_5^{-1} x_2 x_5^{-1} x_2 x_7 x_5 \qquad (2).$$

Therefore

$$x_1 = x_5^{-1} x_2 x_5^{-1} x_2 x_5^{-1} x_2 x_5^{-1} \qquad (3).$$

So $\quad x_4^{-1} = x_2 x_5^{-1} x_2 x_5^{-1} x_2 x_5^{-1} x_2 x_5^{-1} x_2$

and $\quad x_7 = x_5^{-1} x_2 x_5^{-1} x_2 x_5^{-1} x_2 x_5^{-1} x_2 x_5^{-1} x_2 x_5^{-1}$

$$= x_1 x_2 x_5^{-1} x_2 x_5^{-1} \qquad \text{(using (3))}.$$

$$= x_2^{-1} x_4^{-1} x_5^{-1} x_2 x_5^{-1}$$

$$= x_2^{-1} x_5 x_7 x_2 x_5^{-1} \qquad \text{i.e. } x_2 x_7 x_5 = x_5 x_7 x_2.$$

Now *(2)* becomes $x_4^{-1} x_1 x_4^{-1} = x_2 x_7 x_2 x_7 x_2 = x_2 x_6^{-1} x_2$. So $x_6^{-1} = x_1 x_2 x_1 x_2 x_1 = x_1 x_4^{-1} x_1$ giving

$$x_4 = x_1 x_6 x_1 \qquad (4).$$

This means that $x_4 x_6 = x_1 x_5^{-1}$ or $x_4 x_6 x_5 = x_1$ which is $x_3^{-1} x_4^{-1} x_5 = x_1$ or $x_5 = x_4 x_3 x_1$ and so $x_5 x_7 x_1 = x_4$. Applying *(4)* it is seen that $x_5 x_7 = x_1 x_6$, so $x_4^{-1} = x_1 x_6 x_5$ but this is x_6^{-1}. So the group is trivial.

(6.4) <u>The case (1,-1,1,-1).</u>

<u>Lemma 7</u>

Some results about Fibonacci numbers (f_i is the ith Fibonacci number where $f_0=0, f_1=1$, $f_{i+2}=f_{i+1}+f_i$ and g_n is the nth Lucas number ($g_0=2$, $g_1=1$, $g_{i+2}=g_{i+1}+g_i$)).

1: $(f_n, f_{n+1}+1) = (f_{m+2}, f_{m-1})$, when $n=2m+1$,

2: $(f_a, f_b) = f_{(a,b)}$,

3: $(f_{n+1}+1)(f_{n-1}+1)-f_n^2=g_n$ when n is odd.

Proof :-

1 (Taken from [3,Thm. 2.3])

$(f_n,f_{n+1}+1)=(f_{2m+1},f_{2m+2}+1)=(f_{2m+1},f_{2m+1}+f_{2m}+1)=(f_{2m+1},f_{2m}+1)=$ $(f_{2m}+f_{2m-1},f_{2m}+1)=(f_{2m-1}-1,f_{2m}+1)$. By induction it is seen that $(f_{2m+1},f_{2m+2}+1) = (f_{2m-k}+(-1)^k f_{k+1},f_{2m-k+1}-(-1)^k f_k)$. Taking k=m gives the result $(f_{2m+1},f_{2m+2}+1)= (f_m \pm f_{m+1},f_{m+1}\pm - f_m)=(f_{m+2},f_{m-1})$.

2

Firstly by induction show that $f_{k+r}=f_r f_{k+1}+f_{r-1}f_k$. Therefore $(f_{k+r},f_k)=(f_r f_{k+1},f_k)=(f_r,f_k)$ as clearly $(f_{k+1},f_k)=1$. Then induction using this on the maximum of {a,b} gives that (let a>b) $(f_a,f_b) = f_{(a-b,b)}=f_{(a,b)}$.

3

Firstly note that $(-1)^n=f_{n+1}f_{n-1}-f_n^2$ which is easily seen by induction as $f_{n+1}f_{n-1}-f_n^2 = f_nf_{n-1}+f_{n-1}^2-f_n^2 = f_{n-1}^2-f_nf_{n-2}$. Then note that $(f_{n+1}+1)(f_{n-1}+1)-f_n^2 = f_{n+1}f_{n-1}-f_n^2+ f_{n+1}+f_{n-1}+1= (-1)^n+1+f_{n+1}+f_{n-1}$. It is easily seen that $f_{n+1}+f_{n-1} = g_n$ and noting that n is odd completes the proof.

Corollary 7.1

$(f_n,f_{n+1}+1) = f_{(m+2,m-1)}$, when n=2m+1.

The groups to be considered are

$G_n= \langle a,b \mid a^3=1,\ b^n=1,\ aba^{-1}b^2ab^{-2}a^{-1}b^{-1} = 1\rangle$. The orders of all these groups will be determined.

Lemma 8

G_n has the group $X_n = \langle x,y,z \mid x^n=1,\ y^n=1,\ z^n=1,\ yx^2=xy^2,\ zy^2=yz^2,\ xz^2=zx^2\rangle$ as a subgroup of index 3.

Proof :-

Carry out coset enumeration with the subgroup $X_n = \langle x,y,z \rangle$ where $x=b$, $y=aba^{-1}$ and $z=a^2ba^{-2}$. Define cosets $1\ a = 2$ and $2\ a = 3$. Then $a^3=1$ gives that $3\ a = 1$ and the generators give that $1\ b = x\ 1$, $2\ b = y\ 2$ and $3\ b = z\ 3$, the other relations giving the required presentation.

Now when n is odd these groups have already been investigated in [10]. In the notation used in that paper $X_n = S_3G_n$ and the result and proof are as follows, slightly simplified as the paper deals with a more general result.

Theorem 9

If n is odd then S_3G_n has order ng_n^2 if n is coprime to 3 or $2ng_n^2$ otherwise.

Proof :-

Define $m: S_3G_n \rightarrow S_n$ by $x,y,z \rightarrow (1\ 2\ \dots\ n)$ and let $H=\mathrm{Ker}(m)$. Now H is a subgroup of index n in S_3G_n and so has tranversal $U=\{1, x \dots x^{n-1}\}$. Using the Reidemeister-Schreier method we get generators $y_i=x^iyx^{-i-1}$ and $z_i=x^izx^{-i-1}$ and the relations conjugated by U are $x^{i+1}y^2x^{-2}y^{-1}x^{-i} = y_{i+1}y_{i+2}y_i^{-1}$ and $x^{i+1}z^2x^{-2}z^{-1}x^{-i} = z_{i+1}z_{i+2}z_i^{-1}$ with $x^iyz^2y^{-2}z^{-1}x^{-i} = y_iz_{i+1}z_{i+2}y_{i+2}^{-1}y_{i+1}^{-1}z_i^{-1}$. The relations $y^n=1$ and $z^n=1$ give $y_0y_1 \dots y_{n-1} = 1$ and $z_0z_1 \dots z_{n-1} = 1$ respectively. Note that the third relation is $y_iz_i = z_iy_i$. Consider the subgroup $\langle y_i \rangle$. This is clearly generated by y_0 and y_1 but these commute as follows. Apply $y_i = y_{i+1}y_{i+2}$ to $y_0y_1 \dots y_{n-1} = 1$ to get $y_0^2y_2y_4 \dots y_{n-3} = 1$ then substitute $y_{n-3} = y_{n-4}y_{n-2}^{-1}$, $y_{n-5} = y_{n-6}y_{n-4}^{-1}$ and $y_{n-7} = y_{n-8}y_{n-6}^{-1}$ to get $y_0^2y_1y_{n-2}^{-1} = 1$, but $y_{n-2} = y_{n-1}y_0$ and $y_{n-1} = y_0y_1$ so $y_0y_1 = y_1y_0$ and

$\langle y_i \rangle$ is abelian. An inductive argument using this and $y_i = y_{i-2}y_{i-1}^{-1}$ gives that $y_i = y_0^{a(i)}y_1^{b(i)}$ where $a(i)=(-1)^{i+2}f_{i-1}$ and $b(i) = (-1)^{i+1}f_i$. Using the fact that n is odd we see that $y_0^{a(n-2)}y_1^{b(n-2)} = y_{n-2} = y_0^2y_1$ and $y_0^{a(n-1)}y_1^{b(n-1)} = y_{n-1} = y_0y_1$ which gives the relation matrix

$$\begin{bmatrix} a(n-2)-2 & b(n-2)-1 \\ a(n-1)-1 & b(n-1)-1 \end{bmatrix} = \begin{bmatrix} -f_{n-3}-2 & f_{n-2}-1 \\ f_{n-2}-1 & -f_{n-1}-1 \end{bmatrix} = \begin{bmatrix} f_{n+1}+1 & f_n \\ f_n & f_{n-1}+1 \end{bmatrix}$$

Notice, using Corollary 7.1, that $(f_n,f_{n+1}+1)$ is either 3 if $n \equiv 1 \pmod 3$ or 1 otherwise.

Case 1.

$(n,3) = 1$:- Here $(f_n,f_{n+1}+1)=1$ so $\langle y_i \rangle$ is cyclic and $y_iz_i = z_iy_i$ shows that the group H is abelian. Now $| \langle y_i \rangle| = (f_{n+1}+1)(f_{n-1}+1)-f_n^2=g_n$ from Lemma 7. So H is the direct product of 2 cyclic groups of order g_n and G is of order $2g_n^2$.

Case 2.

When $(n,3)=3$:- Here $(f_n,f_{n+1}+1)=2$. Notice that $H = \langle y_0,y_1,z_0,z_1 \rangle$. Since $[y_2^{-1},z_2]=1$, $y_2=y_0y_1^{-1}$ and $z_2=z_0z_1^{-1}$ we have

$$y_0^{-1}y_1z_0z_1^{-1}y_1^{-1}y_0z_1z_0^{-1}=1.$$

Therefore observing that y_0 and y_1 commute, z_0 and z_1 commute, y_0 and z_0 commute and y_1 and z_1 commute we obtain $y_0^{-1}z_1^{-1}y_0z_1y_1^{-1}z_0^{-1}y_1z_0=1$ which is $[y_0,z_1][y_1,z_0]=1$ but $[y_0,z_1]$ commutes with z_0 and y_1 and $[y_1,z_0]$ commutes with z_1 and y_0. Since $[y_0,z_1]=[y_1,z_0]^{-1}$ and $H=\langle y_0,y_1,z_0,z_1 \rangle$ we see that $N=\langle [y_0,z_1] \rangle$ is central in H. Now H/N is $\langle y_0,\ldots,y_{n-1},z_0,\ldots,z_{n-1} \mid y_i=y_{i+1}y_{i+2}, y_0y_1\cdots y_{n-1}=1,$

$z_i=z_{i+1}z_{i+2}$, $z_0z_1 \dots z_{n-1}=1$, $[a,b]=1$ where a,b are any elements of the group $\rangle$. This is because for H/N the relation $[y_0,z_1]$ is added to the relations for H so the generators of $H = \langle y_0,y_1,z_0,z_1 \rangle$ are seen to commute and H is abelian. So it is seen that H/N is isomorphic to $\langle y_i\rangle\times\langle z_i\rangle$ and is therefore of order g_n^2. Finally we show that $[y_0,z_1]^2=1$. For $y_0^{-1}z_1^{-1}y_0z_1y_0^{-1}z_1^{-1}y_0z_1 = y_0^{-2}z_1^{-1}y_0^2z_1$.

But because $(f_n,f_{n+1}+1)=2$ it is seen that y_0^2 is a power of y_1 which commutes with z_1 so N has order 2 giving the original group order $2g_n^2$ as required.

When n is even add the relation z=y to obtain the homomorphic image $X_n^h = \langle x,y \mid x^n=1,\ y^n=1, yx^2=xy^2 \rangle$ which is the group H(2,n) from [11]. Notice as in [4] that the relation $y^n=1$ is redundant as $y^n = (y^2)^{n/2} = (y^{-2}x)^{-1}(x^2)^{n/2}(y^{-2}x)$. If the group is factored by its derived subgroup the factor group is C_n so x is not in the derived group which then has $\{ 1, x, \dots x^{n-1} \}$ as a tranversal. On abelianising the relations of H(2,n), $yx^2=xy^2$ becomes x=y, so yx^{-1} is in the derived group and x^iy has coset representative x^{i+1}, giving subgroup generators $\{x^iyx^{-i-1} = a_i\}$. The relations are $x^{i+1}y^2x^{-2}y^{-1}x^{-i} = a_{i+1}a_{i+2}a_i^{-1}$ so the derived group is the Fibonacci group F(2,n) (see [6]). It is now noted, as in [4], that F(2,n) is infinite for n even and greater than 4.

This leaves the cases n=2 and n=4.

Corollary 9.1

$X_2= C_2$

Corollary 9.2

X_4 is of order 100.

Proof :-

$X_4 = \langle x,y,z \mid x^4=1,\ y^4=1,\ z^4=1,\ yx^2=xy^2,\ zy^2=yz^2,\ xz^2=zx^2 \rangle$. Carry out coset enumeration with the subgroup $\langle x,y \rangle$ defining $i\ z = i+1$ for i from 1 to 3. Then $z^4=1$ gives that $4\ z = 1$. Define $3\ x = 5$ and completion occurs giving, after removing redundant relations, $\langle a,b \mid a^4=1,\ b^4=1,\ ba^2=ab^2 \rangle$. Now use the process described above to show that $F(2,4)$ has index 4 and note that $F(2,4)$ is C_5 [6,Chapter 16].

(6.5) The cases (0,e,f,0).

Firstly we will consider the case obtained when $e=f$. This is the group $\langle a,b \mid a^3=1,\ b^n=1,\ ab^2a=b^2 \rangle$.

Theorem 10

The group $\langle a,b \mid a^3=1,\ b^n=1,\ ab^2a=b^2 \rangle$ is C_n when n is odd or 2 and infinite otherwise.

Proof :-

If n is odd $ab^i=b^2a^{-1}b^{i-2}=b^4ab^{i-4}$ .... So $ab^{n+1}=b^{n+1}a^j$ where $j=(-1)^{(n+1)/2}$ giving $ab=ba$ when $(n+1)/2$ is even and so $ab^2a=b^2$ and then $a^3=1$ implies that $a=1$. When $(n+1)/2$ is odd we have that $ab=ba^{-1}$ so $ab\ ba^{-1}=ba^{-1}ab$ which with $ab^2a=b^2$ and then $a^3=1$ again implies that $a=1$.

When n is 2 the group is C_2. For other even n let $c=3n/2$. If $n/2 \geq 2$ and is also even define $A=(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$.. . and

$$
\begin{array}{lllllllllllll}
B = (& 1 & c+1 & 6 & c+6 & 7 & c+7 & 12 & c+12 & \ldots & c & 2c) \\
(& 2 & 2c+1 & 5 & 2c+6 & 8 & 2c+7 & 11 & 2c+12 & \ldots & c-1 & 3c) \\
(& 3 & 3c+1 & 4 & 3c+6 & 9 & 3c+7 & 10 & 3c+12 & \ldots & c-2 & 4c) \\
(& c+2 & 4c+1 & c+5 & 4c+6 & & \ldots & & & & &) \\
(& c+3 & 5c+1 & c+4 & 5c+6 & & \ldots & & & & &) \\
(& 2c+2 & 6c+1 & 2c+5 & 6c+6 & & \ldots & & & & &) \\
(& 2c+3 & 7c+1 & 2c+4 & 7c+6 & & \ldots & & & & &)
\end{array}
$$

....

Now $A^3=1$, $B^n=1$ and $AB^2A=B^2$ but AB is of infinite order (consider the cycle containing 1).

If $n/2$ is odd then take

$$
\begin{aligned}
A = {} & (1\ 2\ 3)(4\ 5\ 6) \ \ldots\ (c\text{-}5\ \ c\text{-}4\ \ c\text{-}3)(c\text{-}2\ \ c\text{-}1\ \ c)^{-1} \\
& (c+1\ \ c+2\ \ c+3) \ \ldots\ (2c\text{-}5\ \ 2c\text{-}4\ \ 2c\text{-}3)(2c\text{-}2\ \ 2c\text{-}1\ \ 2c)^{-1} \ldots
\end{aligned}
$$

and

$$
\begin{array}{lllllllllllll}
B = (& 1 & c+1 & 6 & c+6 & 7 & c+7 & 12 & c+12 & \ldots & c-2 & 2c-2) \\
(& 2 & 2c+1 & 5 & 2c+6 & 8 & 2c+7 & 11 & 2c+12 & \ldots & c-1 & 3c-2) \\
(& 3 & 3c+1 & 4 & 3c+6 & 9 & 3c+7 & 10 & 3c+12 & \ldots & c & 4c-2)
\end{array}
$$

... and again AB is infinite.

Now the case with $e=-f$ will be considered.

Proposition 11

The group $\langle a,b \mid a^3=1,\ b^n=1,\ ab^2=b^2a \rangle$ is $C_3 \times C_n$ when n is odd or infinite otherwise.

Proof :-

If n is even then add the relation $b^2=1$ to get the free product of cyclic groups of order 2 and 3. If n is odd then

$(b^2)^{(n+1)/2} = (ab^2a^{-1})^{(n+1)/2}$ giving that the group is abelian.

(c) The groups $\left({}_q{}^1{}_r{}^1{}_s{}^{-1}{}_t{}^{-1}\right)$.

Let $q=e_1$, $r=e_2$, $s=e_3$ and $t=e_4$. Let the group $\left({}_q{}^1{}_r{}^1{}_s{}^{-1}{}_t{}^{-1}\right)$ be (e_1, e_2, e_3, e_4).

Notice that (e_1, e_2, e_3, e_4) is isomorphic to $(-e_1, -e_2, -e_3, -e_4)$ and also to (e_1, e_4, e_3, e_2). So when all the e_i are non-zero, the cases to consider are (1,1,1,1), (1,1,-1,-1) and (1,-1,1,-1).

(When n=4 it is seen that the groups (1,1,1,-1) and (-1,1,1,1) are different and (1,1,-1,1) and (1,-1,1,1) are also different using the Todd-Coxeter program.)

No cases exist with just one of the e_i zero. If two are zero then we need to consider (0,0,e,f), (0,e,f,0) and (e,0,0,f). If three are zero then a=1 and the group is C_n and if four are zero, then the group is infinite.

(c.1) The cases (0,0,e,f),(0,e,f,0) and (e,0,0,f).

These groups are all of the form $\langle a,b \mid a^3=1,\ b^n=1,\ aba^j=b\rangle$ where j is 1 or -1. So the groups are of order 3n or n (see the case (e,0,0,f) in the $\left({}_q{}^1{}_r{}^2{}_s{}^{-2}{}_t{}^{-1}\right)$ section above).

(c.2) The case (1,-1,1,-1).

These are the groups $\langle a,b \mid a^3=1,\ b^n=1,\ aba^{-1}bab^{-1}a^{-1}b^{-1}=1\rangle$. Carry out the modified coset enumeration algorithm to find a presentation of the subgroup $\langle b, aba^{-1}, a^2ba^{-2}\rangle$. Let these generators be x, y and z respectively. Defining 1 a = 2 and 2 a = 3 causes completion of the algorithm and the presentation obtained is

$\langle x,y,z \mid x^n=1, y^n=1, z^n=1, xy=yx, xz=zx, yz=zy\rangle$.

So the original groups are of order $3n^3$.

(c.3) The case (1,1,-1,-1).

These are the groups $G_n = \langle a,b \mid a^3=1,\ b^n=1,\ ababa^{-1}b^{-1}a^{-1}b^{-1}=1\rangle$. First note that the factor group obtained by factorising by the normal closure of $\langle (ab)^2\rangle$ is the von Dyck group $(2,3,n)$ and so G_n is infinite for $n>5$ [1]. To investigate the other cases we proceed as below.

Lemma 12

The group G_n has $A_n = \langle a_1,..,a_n \mid a_i^3=1,\ a_1a_2=a_2a_3=a_3a_4... \rangle$ as a subgroup of index n.

Proof :-

Carry out the modified Todd-Coxeter algorithm to find the index and presentation of the subgroup $\langle a,\ bab^{-1},..,\ b^{n-1}ab^{1-n}\rangle$. Let these generators be $a_1,..,a_n$ respectively. Define the cosets 2 to n by $i\ b = i+1$. The generators and the relation $b^n=1$ give that $i\ a = a_i\ i$ and $n\ b = 1$ and so the algorithm is completed and A_n has index n. The other 2 relations of G_n give the required presentation for A_n.

Proposition 13

The group G_1 has order 3, $|G_2|=18$, $|G_3|=72$, G_4 is infinite and $|G_5|=1800$. The group is infinite otherwise.

Proof :-

Note that the relations $a_3^3=1,...,\ a_n^3=1$ are redundant (as $a_1^3=1 \Rightarrow (a_2a_3a_2^{-1})^3=1$ i.e. $a_3^3=1$ etc.) so remove them and if n is odd $a_2^3=1$ is also redundant so remove it as well.

Also note that

$$a_3 = a_2^{-1}a_i a_2$$

$$a_4 = a_2^{-1}a_1^{-1}a_2 a_1 a_2$$

$$a_5 = a_2^{-1}a_1^{-1}a_2^{-1}a_1 a_2 a_1 a_2$$

$$a_6 = a_2^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_2 a_1 a_2 a_1 a_2$$

and it is easily seen by induction that in general

$a_{2m} = (a_1a_2)^{1-m}a_2(a_1a_2)^{m-1}$ and $a_{2m-1} = (a_1a_2)^{1-m}a_1(a_1a_2)^{m-1}$. So the generators $a_3, .., a_n$ can be removed to obtain the presentation $\langle a_1,a_2 \mid a_1^3=1,\ a_2^3=1,\ a_1a_2 = (a_1a_2)^{1-n/2}a_2(a_1a_2)^{n/2-1}\ a_1 \rangle$ when n is even and $\langle a_1,a_2 \mid a_1^3=1,\ a_1a_2 = (a_1a_2)^{(1-n)/2}a_1(a_1a_2)^{(n-1)/2}\ a_1 \rangle$ when n is odd.

Case 1 : n is even.

Here we have the group $\langle x,y \mid x^3=1,\ y^3=1,\ (xy)^{n/2}=(yx)^{n/2} \rangle$. (Add the relation $(xy)^{n/2}=1$ and let $t=xy$ to get $\langle x,t \mid x^3=1,\ t^{n/2}=1\rangle$ which is infinite provided $n/2 > 1$ as required.) Note that A_2 is $C_3 \times C_3$.

Case 2 : n is odd.

This is the group $\langle x,y \mid x^3=1,\ y(xy)^{(n-1)/2}=(xy)^{(n-1)/2}x \rangle$. (Let $z=xy$ and add the relation $z^n=1$ which gives

$\langle x,y,z \mid x^3=1,\ z^n=1,\ yz^{(n-1)/2} = z^{(n-1)/2}x,\ z = xy \rangle$

or $\langle x,z \mid x^3=1,\ z^n=1,\ x^{-1}z^{(n+1)/2} = z^{(n-1)/2}x \rangle$. The third relation is $(z^{(n-1)/2}x)^2 = 1$. Note that $((n-1)/2,n) = 1$ so we can replace z by $t=z^{(n-1)/2}$ to obtain $\langle x,t \mid x^3=1,\ t^n=1,\ (xt)^2=1 \rangle$ which is the Von Dyck group $(2,3,n)$ and so is infinite for $n>5$.)

Note that A_1 is C_3. For A_3 we use the Reidemeister-Schreier algorithm. When A_3 is abelianised it becomes C_3 and also note that

$y^{-1}x = xyx^{-1}y^{-1} \in A_3'$ so a transversal of the derived group is $\{1, x, x^2\}$. The set of generators of the subgroup is $\{x^3, yx^{-1}, xyx^{-2}, x^2y\}$ (Call these a,b,c,d). The set of relations is $\{ x^3, x^iyxyx^{-1}y^{-1}x^{-1-i}\}$. So $a=1$ and the subgroup is $\langle b,c,d \mid bdc^{-1}=1, cbd^{-1}=1, dcb^{-1}=1\rangle$. Removing d gives $\langle b,c \mid bcb=c, cbc=b\rangle$ which is the quaternion group of order 8. A_5 is $\langle x,y \mid x^3=1, yxyxy=xyxyx \rangle$. This group has order 360. The last relation is $x(xyx)x^{-1}(xyx)^{-1}= y^{-1}x$, so again $y^{-1}x$ is contained in the derived group and a transversal is $\{1, x, x^2\}$. As above the Reidemeister-Schreier method is used to obtain the following presentation for the derived group : $\langle b,c,d \mid bdc=cb, cbd=dc, dcb=bd\rangle$ which is $\langle b,c \mid bc^2b=cbc, cb^2c=bcb\rangle$. This group is perfect since it is trivial when abelianised. It is also of order 120 and so is SL(2,5) : the only perfect group of order 120. (To see that the group is of order 120 carry out coset enumeration on the subgroup $\langle bc, bc^{-1}b\rangle$. The following definitions are made : 1b=2, 3c=2, 1b=4, 4a=5 and 4b=6. These cause completion of the process and the subgroup is of index 6 and has a presentation $\langle x,y \mid yx^3y=x^2, xyx=y \rangle$. The first relation is equivalent to $y^2x^5=1$ $(yx^3=x^2y^{-1} \Leftrightarrow yx^3=xy^{-1}x^{-1} \Leftrightarrow yx^3=y^{-1}x^{-2})$ and it also gives that $y^2=x^5$ $(yx^3=x^2y^{-1} \Leftrightarrow x^{-1}yx^2=x^2y^{-1} \Leftrightarrow x^{-2}yx=x^2y^{-1} \Leftrightarrow x^{-3}y=x^2y^{-1})$. So $y^4=1$. Using the second relation any word in the group can be expressed as a power of x post-multiplied by a power of y. Then $y^2=x^5$ can be used to reduce the power of x to an element of $\{0,1,2,3,4\}$ and $y^4=1$ can then reduce the power of y to be from 0 to 3. So it is seen that there are at most 20 elements of this group. However if any two of these listed elements are the same then $x^i=y^j$ for some $i \in \{0,1,2,3,4\}$ and $j \in \{0,1,2,3\}$. If $i \neq 0$ then y is a power of x and if $i=0$ then $j \neq 0$ so either $j=1,3$ and x is a power of y or $y^2=1$. If this latter case is true then the second relation is $(xy)^2=1$ which with $y^2=1$, $x^5=1$ and $y^2x^5=1$ implies that the group is trivial. Finally note that coset enumeration immediately shows that x has index 2 in the group so the group has order 20.

(c.4) The case (1,1,1,1).

These are the groups $G_n=\langle a,b \mid a^3=1,b^n=1,ababab^{-1}ab^{-1}=1\rangle$. Carry out the coset enumeration algorithm on the subgroup $\langle a,bab^{-1},\ldots,b^{n-1}ab^{1-n}\rangle$. Call these generators $a_1,\ldots,a_n$ respectively.

Lemma 14

G_n has $A_n = \langle a_1,\ldots,a_n \mid a_i^3=1,\ a_ia_{i+2}a_{i+3}a_{i+2}=1\ \rangle$ as a subgroup of index n.

Proof :-

Define cosets by i b = i+1 for i from 1 to n-1 and completion occurs as above with the required presentation resulting.

Theorem 15

A_n is trivial if n is odd, otherwise it has order $3n^2$.

Proof :-

If n is odd then note that $a_1a_2a_3=a_2^{-1}$ and $a_2a_3a_4=a_3^{-1}$. These two relations imply that $a_1a_2a_3a_4$ is $a_2^{-1}a_4$ and also $a_1a_3^{-1}$, so $a_2a_1=a_4a_3$. Continuing with this it is seen that $a_2a_1=a_4a_3=a_6a_5=\ldots=a_3a_2$. So $a_1a_2a_3a_2$ is $a_1a_2^{-1}a_1$ giving that $a_1=a_2^{-1}$ and the group is trivial.

If n is even then again $a_2a_1=a_4a_3=a_6a_5=\ldots a_2a_1\ldots$ and $a_3a_2=a_5a_4=a_7a_6=\ldots a_3a_2\ldots$. Note that

$$a_3 = a_2^{-1}a_1^{-1}a_2^{-1} \text{ so}$$

$$a_4 = a_2a_1a_3^{-1} = a_2a_1a_2a_1a_2$$

$$a_5 = a_3a_2a_4^{-1} = a_2^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1} \quad \ldots$$

Inductively it can be seen that for all $r \geq 1$ $a_{2r}=(a_2a_1)^{2r-2}a_2$ and $a_{2r+1}=a_2^{-1}(a_2a_1)^{1-2r}$. These use many relations and on removing redundant generators the only relations remaining are $a_{n-1}a_na_1a_n=1$ which is $a_2^{-1}(a_2a_1)^{3-n}\,(a_2a_1)^{n-2}a_2\; a_1\;(a_2a_1)^{n-2}a_2=1$ or $(a_2a_1)^n=1$, $a_na_1a_2a_1=1$ which also gives $(a_2a_1)^n=1$ and the $a_i^3=1$ relations. Now $a_{2r}{}^3=((a_2a_1)^{2r-2}a_2)^3$ and $a_{2r+1}{}^3=(a_2^{-1}(a_2a_1)^{1-2r})^3$ which are of the form $((a_2a_1)^{i+1}a_2)^3=1$ or $((a_1a_2)^ia_1a_2^{-1})^3=1$. It will now be shown using an inductive argument that all these are consequences of the case $i=0$ i.e. $(a_1a_2^{-1})^3=1$.

Firstly note that $a_1^{-1}a_2^{-1}$ commutes with (a_1a_2).

$$\begin{aligned} a_1^{-1}a_2^{-1}(a_1a_2) &= a_1a_1a_2^{-1}a_1a_2 \\ &= a_1a_2a_1^{-1}a_2a_2 \\ &= (a_1a_2)(a_1^{-1}a_2^{-1}) . \end{aligned}$$

Now look at

$$\begin{aligned} &((a_1a_2)^ia_1a_2^{-1})^3 \\ &\quad= (a_1a_2)^ia_1a_2^{-1}(a_1a_2)^ia_1a_2^{-1}(a_1a_2)^ia_1a_2^{-1} \\ &\quad= (a_1a_2)^{i-1}\,(a_1a_2)\,a_1a_2^{-1}(a_1a_2)(a_1a_2)^{i-1}a_1a_2^{-1}(a_1a_2)^ia_1a_2^{-1} \\ &\quad= (a_1a_2)^{i-1}\,a_1a_2^{-1}a_1^{-1}a_2^{-1}(a_1a_2)^{i-1}a_1a_2^{-1}(a_1a_2)^ia_1a_2^{-1} \\ &\quad= (a_1a_2)^{i-1}\,a_1a_2^{-1}(a_1a_2)^{i-1}a_1^{-1}a_2^{-1}a_1a_2^{-1}(a_1a_2)^ia_1a_2^{-1} \\ &\quad= (a_1a_2)^{i-1}\,a_1a_2^{-1}(a_1a_2)^{i-1}a_1a_2a_1^{-1}(a_1a_2)^ia_1a_2^{-1} \\ &\quad= (a_1a_2)^{i-1}\,a_1a_2^{-1}(a_1a_2)^{i-1}a_1a_2^{-1}(a_1a_2)^{i-1}a_1a_2^{-1} \\ &\quad= ((a_1a_2)^{i-1}\,a_1a_2^{-1})^3 . \end{aligned}$$

So, calling $a_1=a$ and $a_2=b$, we get the following presentation for

A_n when n is even : $\langle a,b \mid a^3=1,\ b^3=1,\ (ab^{-1})^3=1,\ (ab)^n=1\rangle$. Now carry out the coset enumeration algorithm on the subgroup $\langle ab,\ a^2ba^{-1},\ ab^{-2}\rangle$. Let these generators be called x, y and z respectively. Define $1\,a = 2$ and $2\,a = 3$. It is then seen from $a^3 = 1$ that $3a = 1$ and the generators show that $2b = x1$, $3b = y2$ and $1b=z3$. The other generators give the group $\langle x,y,z \mid zyx=1,\ xyz=1,\ x^n=1,\ y^n=1,\ z^n=1\rangle$. On removing z we get the presentation $\langle x,y \mid xyx^{-1}y^{-1}=1,\ x^n=1,\ y^n=1,\ (yx)^n=1\rangle$ or $C_n\times C_n$.

(d) <u>The groups</u> $\left({}_q{}^2{}_r{}^2{}_s{}^{-2}{}_t{}^{-2}\right)$.

Let $q=e_1$, $r=e_2$, $s=e_3$ and $t=e_4$. The orders of these groups are known as a corollary to the determination of the groups $\left({}_q{}^1{}_r{}^1{}_s{}^{-1}{}_t{}^{-1}\right)$. These are the groups

$\left({}_q{}^2{}_r{}^2{}_s{}^{-2}{}_t{}^{-2}\right)(n) = \langle a,b \mid a^3=1,\ b^n=1,\ a^qb^2a^rb^2a^sb^{-2}a^tb^{-2}=1\rangle$. If n is odd the group is generated by a and b^2 and on renaming the latter b, $\left({}_q{}^1{}_r{}^1{}_s{}^{-1}{}_t{}^{-1}\right)(n)=\langle a,b \mid a^3=1,\ b^n=1,\ a^qba^rba^sb^{-1}a^tb^{-1}=1\rangle$ is obtained. If n is even then as before apply the coset enumeration process to find the presentation of $\langle a,\ bab^{-1},\dots,\ b^{n-1}ab^{1-n}\rangle$ which is normal and of index n. The presentation obtained is

$\langle a_1,\dots,a_n \mid a_i^3=1,\ a_{i+1}{}^pa_{i+3}{}^qa_{i+5}{}^ra_{i+3}{}^s=1,\ i=1,2,\dots,n\ \rangle$ which is the free product of $\langle a_1,a_3,\dots,a_{n-1} \mid a_i^3=1,\ a_i{}^pa_{i+2}{}^qa_{i+4}{}^ra_{i+2}{}^s=1,\ i=1,3,\dots,n-1\ \rangle$ and $\langle a_2,a_4,\dots,a_n \mid a_i^3=1,\ a_{i+1}{}^pa_{i+3}{}^qa_{i+5}{}^ra_{i+3}{}^s=1,\ i=1,3,\dots,n-1\ \rangle$.

However if the same process is applied to $\left({}_q{}^1{}_r{}^1{}_s{}^{-1}{}_t{}^{-1}\right)(n/2)=\langle a,b \mid a^3=1, b^{n/2}=1,\ a^qba^rba^sb^{-1}a^tb^{-1}=1\rangle$ the subgroup, of index n/2,

$\langle a_1,..,a_{n/2} \mid a_i^3=1,\ a_i^p a_{i+1}^q a_{i+2}^r a_{i+1}^s=1,\ i=1,2,...,n\rangle$ is obtained. So it is seen that if n is even then if the group $\left(q^1 r^1 s^{-1} t^{-1}\right)(n/2)$ is $C_{n/2}$ the group $\left(q^2 r^2 s^{-2} t^{-2}\right)(n)$ is C_n otherwise the latter group is infinite. i.e. when n is even

$$\left(q^2 r^2 s^{-2} t^{-2}\right)(n)/\ C_n \equiv$$

$$\left(q^1 r^1 s^{-1} t^{-1}\right)(n/2)/C_{n/2} * \left(q^1 r^1 s^{-1} t^{-1}\right)(n/2)/C_{n/2}$$

and when n is odd

$$\left(q^2 r^2 s^{-2} t^{-2}\right)(n) \equiv \left(q^1 r^1 s^{-1} t^{-1}\right)(n).$$

All the necessary cases have been considered : if q+r+s+t=0 then in each case $\left(q^2 r^2 s^{-2} t^{-2}\right) \equiv \left(t^2 s^2 r^{-2} q^{-2}\right)$ but $\left(q^1 r^1 s^{-1} t^{-1}\right) \equiv \left(t^1 s^1 r^{-1} q^{-1}\right)$ also. If q+r+s+t=3 or -3 then one of {q,r,s,t} is zero and neither of the two isomorphisms hold. If q+r+s+t is not divisible by 3 and n is even then the derived group of $\left(q^2 r^2 s^{-2} t^{-2}\right)(n)$ is $\langle a, bab^{-1},...,\ b^{n-1}ab^{1-n}\rangle$ and so if $\left(q^2 r^2 s^{-2} t^{-2}\right)(n) \equiv \left(t^2 s^2 r^{-2} q^{-2}\right)(n)$ then as the derived groups of isomorphic groups must be isomorphic it is seen that $\langle a_1,..,a_{n/2} \mid a_i^3=1,\ a_i^p a_{i+1}^q a_{i+2}^r a_{i+1}^s=1,\ i=1,2,...,n\ \rangle$ is isomorphic to $\langle a_1,..,a_{n/2} \mid a_i^3=1,\ a_i^s a_{i+1}^r a_{i+2}^q a_{i+1}^p=1,\ i=1,2,...,n\ \rangle$. (Call this isomorphism k). But these are the derived groups of $\left(q^1 r^1 s^{-1} t^{-1}\right)(n)$ and $\left(t^1 s^1 r^{-1} q^{-1}\right)(n)$. This implies an obvious isomorphism between $\left(q^1 r^1 s^{-1} t^{-1}\right)(n)$ and $\left(t^1 s^1 r^{-1} q^{-1}\right)(n)$. Note that the transversal of the derived groups in these is

$\{1,b,\ldots,b^{n-1}\}$. Construct the map from $(q^1 r^1 s^{-1} t^{-1})(n)$ to $(t^1 s^1 r^{-1} q^{-1})(n)$ given as follows. Take an element c of $(q^1 r^1 s^{-1} t^{-1})(n)$. Say that this is in the same coset as b^j. Then map this to $k(cb^{-j})b^j$. Using the fact that the derived group is normal this map is seen to be an isomorphism.

If n is odd then obviously $(q^2 r^2 s^{-2} t^{-2}) \equiv (t^2 s^2 r^{-2} q^{-2}) \Rightarrow$ $(q^1 r^1 s^{-1} t^{-1}) \equiv (t^1 s^1 r^{-1} q^{-1})$.

It is clear that if $(q^1 r^1 s^{-1} t^{-1}) \equiv (t^1 s^1 r^{-1} q^{-1})$ then $(q^2 r^2 s^{-2} t^{-2}) \equiv (t^2 s^2 r^{-2} q^{-2})$. Again this is immediate if n is odd. If n is even then there exists an isomorphism between

$(q^1 r^1 s^{-1} t^{-1})(n/2)/C_{n/2} * (q^1 r^1 s^{-1} t^{-1})(n/2)/C_{n/2}$ and

$(t^1 s^1 r^{-1} q^{-1})(n/2)/C_{n/2} * (t^1 s^1 r^{-1} q^{-1})(n/2)/C_{n/2}$ and so let k be the associated isomorphism between

$\langle a_1,..,a_n \mid a_i^3=1,\ a_{i+1}{}^q a_{i+3}{}^r a_{i+5}{}^s a_{i+3}{}^t=1, i=1,2,\ldots,n\ \rangle$ and

$\langle a_1,..,a_n \mid a_i^3=1,\ a_{i+1}{}^t a_{i+3}{}^s a_{i+5}{}^r a_{i+3}{}^q=1, i=1,2,\ldots,n\ \rangle$. Let the map

$k^\wedge: (q^2 r^2 s^{-2} t^{-2}) \to (t^2 s^2 r^{-2} q^{-2})$ take c, contained in the same coset as b^j, to $k(cb^{-j})b^j$. Because the $\langle a_1,..,a_n\rangle$ subgroups are normal in $(q^2 r^2 s^{-2} t^{-2})$ and $(t^2 s^2 r^{-2} q^{-2})$,(seen from the coset table), this map is an isomorphism. ($k(cb^{-j})b^j k(c_1 b^{-g})b^g = k(cb^{-j})k(c_1 b^{-g})b^{j+g} = k(cb^{-j}c_1 b^{-g})b^{j+g} = k(cc_1 b^{-j-g})b^{j+g}$).

So if 3 does not divide q+r+s+t then $\left({}_{q}{}^{2}{}_{r}{}^{2}{}_{s}{}^{-2}{}_{t}{}^{-2}\right) \equiv \left({}_{t}{}^{2}{}_{s}{}^{2}{}_{r}{}^{-2}{}_{q}{}^{-2}\right) \Leftrightarrow \left({}_{q}{}^{1}{}_{r}{}^{1}{}_{s}{}^{-1}{}_{t}{}^{-1}\right) \equiv \left({}_{t}{}^{1}{}_{s}{}^{1}{}_{r}{}^{-1}{}_{q}{}^{-1}\right)$ and all the cases with the property that $\left({}_{q}{}^{2}{}_{r}{}^{2}{}_{s}{}^{-2}{}_{t}{}^{-2}\right) \equiv \left({}_{t}{}^{2}{}_{s}{}^{2}{}_{r}{}^{-2}{}_{q}{}^{-2}\right)$ have already been considered in the section on the $\left({}_{t}{}^{1}{}_{s}{}^{1}{}_{r}{}^{-1}{}_{q}{}^{-1}\right)$ groups.

Chapter 3 : The H(a,b,c) Groups

(a) Introduction

In [2], the $F(a,b,c) = \langle R,S \mid R^2=1,\ RS^aRS^bRS^c=1 \rangle$ groups were considered. To help determine the order of these, the homomorphic image $H(a,b,c) = \langle R,S \mid R^2=1,\ RS^aRS^bRS^c=1,\ S^{2n}=1,\ n=a+b+c \rangle$ were also looked at. A formula for the orders of the $H(a,b,c)$ groups was obtained and then, in special cases when $c=xa+yb$ with x, y fixed, a simpler formula for the order was obtained. We shall attempt to use Maple programs to find formulas for new special cases.

(b) The general formula

This is taken from [2].

Theorem 1 If $n>0$ then :

If $(a,b,c)=1$ and a,b,c are not all congruent modulo 6 then

$|H(a,b,c)| = 2n|\det(A)|$ where A is the nXn matrix $A_{i,j} = \delta_{i,j} - \delta_{i+a,j} + \delta_{i+a+b,j}$ (if $i \equiv j \pmod n$ then $\delta_{i,j} = (\ n-(j \bmod (2n))+1/2\ ) / |\ n-(j \bmod (2n)\ ) +1/2\ |$ with $\delta_{i,j}=0$ otherwise). If $(a,b,c)=1$ and $a \equiv b \equiv c \pmod 6$ then $H(a,b,c)$ is infinite. If $(a,b,c)=t \neq 1$ then $H(a,b,c)$ is infinite unless $H(a/t,b/t,c/t)=C_{2n/t}$ in which case $H(a,b,c)= C_{2n}$.

Sketch of proof :-

The derived group of $H(a,b,c)$ is

$K(a,b,c)= \langle X_1,\ldots X_n \mid Y_{i+a}=Y_iY_{i+a+b}, 1 \leq i \leq 2n \rangle$, where

$Y_i = X_i^{(n-(i \bmod (2n))+1/2)/|n-(i \bmod (2n))+1/2|}$, which is of index 2n. If (a,b,c)=1 then this subgroup is abelian whereas if (a,b,c)=t then it is a free product of t copies of K(a/t,b/t,c/t) and so is infinite if the latter is non-trivial (so if (a,b,c)=t>1 then |H(a,b,c)|=2n if |H(a/t,b/t,c/t)|=2n/t or infinite otherwise). It was then shown that K(a,b,c) is infinite iff $a \equiv b \equiv c \bmod 6$. When (a,b,c)=1 the fact that K(a,b,c) is abelian allows us to write down the given order matrix.

If n=0 then add R=1 to H(a,b,c) to obtain the free group $\langle S| \;\rangle$. If n<0 note that H(a,b,c) is isomorphic to H(-a,-b,-c). So it is seen that the order of H(a,b,c) can always be found using the above theorem.

(c) Known results

The results obtained in [2] are as follows :

Note in every case (a,b)=1, otherwise the groups would be infinite or C_{2n}. Also in [2] the order of H(a,b,c) is not always given but this is very easy to obtain.

c=-2a-2b H(a,b,-2a-2b)=C_{2n}

The paper shows that F(a,b,-2a-2b)=C_{2n} when 5 does not divide a-b so, as H(a,b,c) is at least order 2n, H(a,b,-2a-2b)=C_{2n} also. When $a \equiv b \pmod 5$ add the relation corresponding to $S^{2n}=1$ (i.e. A=1) to the presentation for F(a,b,-2a-2b) given in [2] to obtain C_{2n} again.

c=2a |H(a,b,c)|=2n|det(A-B)| where

$$A=\begin{vmatrix} x_{n-2} & x_{n-1} & x_n \\ x_{n-1} & x_n & x_{n+1} \\ x_n & x_{n+1} & x_{n+2} \end{vmatrix} \qquad B=(-1)^a \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$x_{m+3}=x_{m+2}-x_m \qquad x_1=0, x_2=1, x_3=0$$

c=a+b $|H(a,b,a+b)|=4(a+b)(2^{a+b}+1-2^{1+1/2(a+b)}\cos 1/4(a-b)\pi)$

Note that this is based on a recurrence relation.

c=-b $|H(a,b,-b)|=2a(2^a-(-1)^b)$

c=-2a $|H(a,b,-2a)|=2n(f_{n+1}+f_{n-1}+1+(-1)^n)$ where f_n is the nth Fibonacci number.

c=b If $|a-b|=1$ then $H(a,b,b)=C_{2n}$. This is the only result for H(a,b,b) given in the paper. However it is easy to obtain the result in general. As in the group F(a,b,b) in the paper note that $RS^aRS^bRS^b=1 \Rightarrow S^{b-a}=(S^bR)^3$ and so $S^{b-a}\leftrightarrow R$. But in H(a,b,b) $S^{2n}=1$ with $2n=2a+4b$ and so $S^{b-a}S^{b-a}S^{2n}=S^{6b}\leftrightarrow R$ as does S^{6a} also. Noting that $(a,b)=1$ we see that $S^6\leftrightarrow R$.

When $b-a \equiv 1,5 \pmod 6$ then using $S^{b-a}\leftrightarrow R$ and $S^6\leftrightarrow R$ we see that H(a,b,b) is abelian. When $b-a \equiv 2,4 \pmod 6$ then $S^2\leftrightarrow R$ and as b-a is even and (a,b)=1, a and b must both be odd. Notice that $RS^aRS^bRS^b=1 \Leftrightarrow S^2RS^{-1}RS^{-1}RS^{(a-1)+(b+1)+b}=1$ or $S^2RS^{-1}RS^{-1}RS^n=1$. Carry out modified coset enumeration on the subgroup $\langle R,SRS^{-1},\ldots,S^{n-1}RS^{1-n}\rangle$. Let these generators be called $a_1,\ldots,a_n$ respectively. Define $1\ S=2,\ldots,\ n-1\ s=n$ and then the subgroup generators give that $i\ R=a_i\ i$ for i from 1 to n. The relation $R^2=1$ shows that $a_i^2=1$ and $1\ S^2RS^{-1}RS^{-1}RS^n=1$ shows that $a_3a_2a_1\ n\ S=1$ ie $n\ S=a_1^{-1}a_2^{-1}a_3^{-1}\ 1$. Thus we have completion. The relation $i\ S^2RS^{-1}RS^{-1}RS^n=i$ for i from 2 to n-2 is $a_{i+2}a_{i+1}a_i=a_3a_2a_1$, $n-1\ S^2RS^{-1}RS^{-1}RS^n=n-1$ is $a_1^{-1}a_2^{-1}a_3^{-1}a_1a_3a_2a_1a_na_{n-1}a_1^{-1}a_2^{-1}a_3^{-1}=1$ and $n\ S^2RS^{-1}RS^{-1}RS^n=n$ is $a_1^{-1}a_2^{-1}a_3^{-1}a_2a_1a_3a_2a_1a_na_1^{-1}a_2^{-1}a_3^{-1}=1$. The relation $RS^2RS^{-2}=1$ together with $a_i^2=1$ shows that

$$a_1=a_3=\ldots=a_n \quad \text{and} \quad a_2=a_4=\ldots=a_{n-1} \qquad (*)$$

(note that n is odd as a is odd). The n-1 and n terms give $a_{n-1}a_1^{-1}a_2^{-1}a_3^{-1}a_1a_3a_2a_1 = 1$ and $a_na_1^{-1}a_2^{-1}a_3^{-1}a_2a_3a_2a_1 = 1$ and the relation $S^{2n}=1$ gives $(a_3a_2a_1)^2=1$. Using (*) we see that the group is the von-Dyck group (2,2,3). So H(a,b,b) is seen to have order 6n.

When b-a $\equiv 3 \pmod 6$ then $S^3 \leftrightarrow R$ and as b-a is divisible by 3 and (a,b)=1 it is seen that $a=3\alpha\pm1$ and $b=3\beta\pm1$ for some α and β. In both of these cases it is therefore seen that a presentation for H(a,b,b) is $\langle R,S \mid R^2=1,\ S^2RS^{-1}RS^{-1}S^n=1, S^{2n}=1,\ S^3\leftrightarrow R \rangle$ so apply the above method. This time $S^3 \leftrightarrow R$ enables us to reduce the number of subgroup generators to 3 and on using this its presentation is seen to be that for $C_2 \times C_2 \times C_2$ and the order of H(a,b,b) is 8n.

When b-a $\equiv 0 \pmod 6$ then $S^6 \leftrightarrow R$ is all that can be deduced. As b-a is divisible by 6 and (a,b)=1 it is seen that $a=6\alpha\pm1$ and $b=6\beta\pm1$ for some α and β. So $\langle R,S \mid R^2=1,\ S^2RS^{-1}RS^{-1}S^n=1,\ S^{2n}=1,\ S^6\leftrightarrow R \rangle$ is now obtained for the presentation of H(a,b,b). As before carry out the modified coset enumeration algorithm on the subgroup $\langle R, SRS^{-1},\ldots, S^{n-1}RS^{1-n}\rangle$. This is as before except that the commutative relation now produces $a_1 = a_7 = \ldots = a_{n-2}$, $a_2 = a_8 = \ldots = a_{n-1}$, $a_3 = a_9 = \ldots = a_n$, $a_4 = a_{10} = \ldots = a_{n-5}$, $a_5 = \ldots = a_{n-4}$ and $a_6 = \ldots = a_{n-3}$. If the 4 following relations : $a_1 = a_4$, $a_3 = a_2$ and $a_5 = a_1a_2a_1 = a_6$ are added then this homomorphic image of the subgroup is seen to be the free product of C_2 with C_2 and hence H(a,b,b) is infinite in this case.

<u>**c=2a+b**</u> |H(a,b,2a+b)|=2n|det(A+B)| where

$$A=\begin{vmatrix} x_k & x_{k+1} & x_{k+2} \\ x_{k+1} & x_{k+2} & x_{k+3} \\ x_{k+2} & x_{k+3} & x_{k+4} \end{vmatrix}$$

k is the integral part of n/2.

$$B=(-1)^{(a+1)/2} \begin{vmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

if a is odd.

$$B=(-1)^{a/2} \begin{vmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

if a is even.

$$x_{m+3}=2x_{m+2}-x_{m+1}-x_m \qquad x_0=1, x_1=-1, x_2=0$$

(d) Maple Programs

It was noted that many of the above results are fixed by n and the parity of a and b and for fixed parities of a and b the order of H(a,b,la+mb) as a and b vary is a recurrence relation depending on n. Some Maple programs were written to attempt to find new results of the same type. The following are the contents of a file in Maple. It is convenient to keep all the programs together as then they can be entered by just selecting the whole file : the programs are then ready to run.

Comments are given in italics and obviously do not appear in the file.

```
f:=proc(a,b,c);
```

```
n:=a+b+c;
d:=array(1..n,1..n);
for i from 1 to n do
if -i mod 2*n<n then e[1]:=-1 else e[1]:=1 fi;
if -(i+a) mod 2*n <n then e[2]:=1 else e[2]:=-1 fi;
if -(i+a+b) mod 2*n <n then e[3]:=-1 else e[3]:=1 fi;
for j from 1 to n do
if i-j mod n = 0 then d[i,j]:=e[1] else d[i,j]:=0 fi;
if (i+a)-j mod n = 0 then d[i,j]:=d[i,j]+e[2] fi;
if (i+a+b)-j mod n = 0 then d[i,j]:=d[i,j]+e[3] fi;
od;
od;
t:=2*n*det(d);
abs(t);
end;
```

The above procedure works out 2n|det(A)| *assuming* n>0. *Note that the mod function in Maple produces a result in the range* 0,...,2n-1 *whereas we have two cases :- where* 1≤j mod (2n)≤n *and* n+1≤j mod(2n)≤2n. *So to convert this we use* -j mod (2n) *etc.*

```
inf:=proc(x,y,z);
print(`H(`,x,y,z,`) has infinite order.`);
0
end;


startvalues:=proc(x,y);
if (x+1)<>0 and (y+1)<>0 then
gg:=gcd(x+1,y+1); gxx:=(x+1)/gg; gyy:=(y+1)/gg;
```

```
g1:=1;
while gxx*g1 mod gyy <>1 mod gyy do
g1:=g1+1 od;
g2:=(gg-(x+1)*g1)/(y+1);
else if y+1=0 and x+1=0 then print (`Order is always infinite.`);
  ge:=2 else
g1:=1;g2:=1;if y+1=0 then gyy:=0;gxx:=1 else gyy:=1;gxx:=0 fi fi fi
end;
```

The above procedure finds values of a *and* b *for which* n *is as small as possible. Note that for large* n *the program* f *is very slow, and in fact the speed of the program increases very rapidly relative to* n. *Clearly the smallest possible value of* n *is* (x+1,y+1). g1 *and* g2 *are values of* a *and* b *with this* n. *All other values with the same* n *are of the form* (g1+λgyy,g2-λgxx).

```
gupdate:=proc(x,y,no,nu,v);x1:=x;y1:=y;print(`A`,`B`,x1,`*A+`,y1,`*B`);
ge:=0;
startvalues(x,y);
gn:=0;
gen:=0;gea:=1;geb:=1;
if gyy mod 2=1 then gea:=0 fi;if gxx mod 2 =1 then geb:=0 fi;
if gea*geb=1 then while ge=0 and gn<nu+1 do
gn:=gn+1;gi:=gn*g1;gj:=gn*g2;
h(ge,gi,gj,gyy,gxx,x1,y1,no,1);ge:=hge od else gn:=1;
if gea=1 then giparity[1]:=1;giparity[0]:=1 else
giparity[0]:=0;giparity[1]:=0  fi;
if geb=1 then gjparity[1]:=1;gjparity[0]:=1 else
gjparity[0]:=0;gjparity[1]:=0  fi;
```

```
while ge=0 and gn<2*nu+1 do
gn:=gn+2;gi:=gn*g1;gj:=gn*g2;
if giparity[gi mod 2]=0 or gjparity[gj mod 2]=0 then
aa:=h(ge,gi,gj,gyy,gxx,x1,y1,no,2) fi;
if giparity[gi+gyy mod 2]=0 or gjparity[gj-gxx mod 2]=0 then
bb:=h(ge,gi+gyy,gj-gxx,gyy,gxx,x1,y1,no,2) fi;
if aa= -1 then giparity[gi mod 2]:=1;gjparity[gj mod 2]:=1;gen:=1 fi;
if bb= -1 then giparity[gi+gyy mod 2]:=1;gjparity[gj-gxx mod 2]:=1;
gen:=1 fi;
if aa=bb then gen:=gen else gen:=1 fi;
if giparity[0]=1 and giparity[1]=1 and gjparity[0]=1 and gjparity[1]=1
then ge:=1 fi od fi;
if ge=1 then print (`Order is independent of n and parities of a and
b.`) else
gt:=array(1..5);for ii from 1 to 5 do gt[ii]:=0 od;
if gen=0 then print(`Order may be function of n.`);gt[1]:=1 else
if giparity[0]=0 then
print(`When a is even, order may be function of n.`);gt[2]:=1 fi;
if giparity[1]=0 then
print(`When a is odd, order may be function of n.`);gt[3]:=1 fi;
if gjparity[0]=0 then
print(`When b is even, order may be function of n.`);gt[4]:=1 fi;
if gjparity[1]=0 then
print(`When b is odd, order may be function of n.`);gt[5]:=1 fi;
fi;
for gii from 1 to 5 do if gt[gii]=1 then supdatedd(v,x,y,gii-1) fi od fi
end;
```

gupdate *is the master program. It first of all tests if the order* *of* H(a,b,c) *depends only on* n *or on* n *and the parity of* a *or maybe* n *and the parity of b. Then if all the checked groups conform to one of the above it then calls* supdatedd *which attempts to fit the orders to a recurrence relation. The flag* gen *has value* 1 *if the order has been found to not depend only on* n *or* 0 *otherwise.* gea *records whether the order may or may not depend on* n *and the parity of* a *and* geb *does the same for* b. nu *is the number of different values of* n *for which the groups are considered,* no *is the number of different groups of each order to be examined and* ge *records whether any relations can be found. Initially* gea *is set to* 1 *if* gyy *is even, as then the parity of* a *is a function of* n *and so if the order is dependent on* n *and the parity of* a, *it is also dependent just on* n. *Similarly for* geb. *If* gea *and* geb *are both* 1 *initially then the only thing to test for is whether the order depends on* n *.Otherwise the program runs through the main* while *loop. Here it tests two strings of groups, each string consisting of groups with* a *and* b *of the same parity.*

```
h:=proc(ge,gi,gj,gyy,gxx,x1,y1,no,m);
gr:=0;ga:=array(1..no);hge:=ge;hgi:=gi;hgj:=gj;
while hge=0 and gr<no do
gr:=gr+1;hgi:=hgi+m*gyy;hgj:=hgj-m*gxx;
if  gcd(hgi,hgj)=1  then  ga[gr]:=ff(hgi,hgj,x1*hgi+y1*hgj);
if ga[gr]<>ga[1] then hge:=1 fi;
else gr:=gr-1 fi; od; if hge=1 then ga[1]:=-1 fi;hga:=ga[1];
hga;
end;
```

The above procedure tests whether no *groups with* n= (x1+1)gi + (y1+1)gj *are of the same order. If they are then it returns their* *order as* hga. *If not then* hge *is set to* 1.

```
choice:=proc(i,sge,g1,g2,gxx,gyy);
oa:=i*g1+gyy;ob:=i*g2-gxx;
if sge =1 or sge =2 then
while ((oa mod (2) <> sge-1) or (gcd(oa,ob)<>1)) do
oa:=oa+gyy;ob:=ob-gxx od else
if sge>2 then while (ob mod 2 <> sge-3 ) or
gcd(oa,ob) <> 1  do oa:=oa+gyy;ob:=ob-gxx od fi fi end;
```

choice *produces values of* a *and* b *which are coprime, of the correct parity and which have the correct* n *value.*

```
supdatedd:=proc(st,x,y,sge);
ti:=0;ste:=0;so:=array(1..(2*st));
startvalues(x,y);tc:=1;
if sge=1 or sge=3 then step:=2 else step:=1 fi;
while ti=0 and ste<st do ste:=ste+1;
choice(2*(ste-1)*step+1,sge,g1,g2,gxx,gyy);soa:=oa;sob:=ob;
so[2*ste-1]:=ff(soa,sob,soa*x+sob*y)/(soa*(x+1)+sob*(y+1));
choice(2*(ste-1)*step+1+step,sge,g1,g2,gxx,gyy);soa:=oa;sob:=ob;
so[2*ste]:=ff(soa,sob,soa*x+sob*y)/(soa*(x+1)+sob*(y+1));
Z:=array(1..ste);m:=array(1..ste,1..ste);
for i from 1 to ste do
Z[i]:=so[ste+i];
for j from 1 to ste do
m[i,j]:=so[ste-j+i]; od od;
vc:=linsolve(m,Z);if vc<>NULL then recrelntest(vc,ste) fi od;
```

```
print(tc);
print(so);
end;
```

supdatedd *attempts to find a recurrence relation of order up to* st *on the groups* H(a,b,c) *with* a *even if* sge=1, a *odd if* sge=2, b *even if* sge=3, b *odd if* sge=4 *or with no reference to the parity of* a *or* b *if* sge=0. *Note that in* choice *an infinite loop can occur. For example, consider when* sge=1. *Note that* oa *is always of the form* $i*g1+gyy*\lambda$ *and* gcd(oa,ob) *is of the form* $(i*g1+gyy*\lambda, i*g2-gxx*\lambda)$. *Now* $g1*gxx+g2*gyy=1$ so

$$(i*g1+gyy*\lambda, i*g2-gxx*\lambda) \mid (g2(i*g1+gyy*\lambda), i*g2-gxx*\lambda)$$

$$= (i*g1*g2+\lambda-g1*gxx*\lambda, i*g2-gxx*\lambda)$$

$$= (\lambda, i*g2-gxx*\lambda)$$

$$= (\lambda, i*g2).$$

Similarly it also divides $(\lambda, i*g1)$ *and hence* (λ, i), *but this clearly divides the original gcd so* $(i*g1+gyy*\lambda, i*g2-gxx*\lambda) = (\lambda, i)$. *Now* $i*g1+gyy*\lambda$ *is odd is equivalent to* i *and* g1 *being odd and* gyy *even. (Note that* gyy *even implies that* g1 *is odd so the latter may be removed). So a loop occurs iff always either* i *is odd and* gyy *even or* $(\lambda, i)\geq 2$. *Of course* i *is fixed so this gives two occasions for a loop : when* i *is odd and* gyy *even or when* i *is even and* gyy *odd. When* sge=2 *a loop occurs iff* i *and* gyy *are both even. Similarly for* sge=3 *we have loops when* i *and* gxx *are of opposite parities and when* sge=4 *the problem is when* i *and* gxx *are both even. However if* gxx *is even then the parity of* b *is fixed for each* n, *so* gupdate *doesn't test for relations with the parity of* b *fixed i.e. when* sge=3 *or* 4 *and similarly when* gyy *is even* choice *isn't called with* sge=1 *or* 2. *So the only 2 occasions for loops are when* sge=1, gyy *is odd and* i *is even and when* sge=3, gxx *is odd and* i *is even hence on those occasions* supdated *ensures that* i *is always odd.*

The if *statement that calls* recrelntest *is essential because if the equation* mx=Z *has no solution (for example see in the section on the output of the programs the output of* gupdate(2,0,6,6,9) *: when* b *is even the first 4 terms are* 2,2,2,16 *so*

$$m=\begin{vmatrix}2 & 2\\ 2 & 2\end{vmatrix} \text{ and } Z=\begin{vmatrix}2\\ 16\end{vmatrix}\quad)$$

then vc *is set to null-value, so calling* recrelntest *would produce an error.*

```
recrelntest:=proc(rvc,rst);ti:=0;ri:=0;tti:=0;
while ri<rst and ti=0 do
ri:=ri+1;if type(rvc[ri],rational)=false then ti:=1 fi od;
if ti=1 then vc:=array(sparse,1..rst-1) else tc:=array(sparse,1..rst)
fi;
tc:=add(tc,vc) end;
```

When the vector vc *produced by* supdatedd *contains a general parameter, then it appears as if a recurrence relation has been found with associated vector* vc *of order* ste. *As this contains a parameter the previous* vc *vector of order* ste-1 *is also a recurrence relation vector and is of minimum length.* recrelntest *returns the value of* 1 *for* ti *when a recurrence relation has been found and also returns the* vc *vector of order* ste-1 *in that case. This is performed by the last* if *statement in the procedure. It isn't possible simply to write* if ti=0 then tc:=vc fi; *which seems natural because once this statement has been passed, the variables* tc *and* vc *have been identified and henceforth changing* vc *automatically changes* tc. *The only way to ensure that when* ti=1, tc *doesn't become* vc, *but in other cases it does : and therefore retain a copy of the previous* vc *vector after a recurrence relation has been found is to use the* if *statement written in the program.*

```
ff:=proc(a1,b1,c1);
n1:=a1+b1+c1;
if n1=0 then inf(a1,b1,c1) else
if n1<0 then a:=-a1;
b:=-b1;
c:=-c1;
n1:=-n1 else a:=a1;
b:=b1;
c:=c1 fi;
if a*b*c=0 then 2*n1 else
u:=gcd(a,gcd(b,c));
if u=1 then
if a mod 6= b mod 6 and a mod 6= c mod 6 then inf(a1,b1,c1) else
f(a,b,c) fi else
if f(a/u,b/u,c/u)=2*n then 2*n1 else inf(a1,b1,c1) fi fi fi fi;
end;
```

The command ff(a,b,c) *outputs the order of the group* H(a,b,c) *according to* Theorem 3.1.

with(linalg);

This is necessary as many of the functions used in the above procedures are contained in the linear algebra package : this command calls this package.

The programs were unable to obtain any new results. However they did confirm many of the known results and a comparison of the orders printed out in two cases leads to a new theorem.

(e) <u>The output of the programs</u>

The programs gave the following results :-

```
gupdate(-2,-2,6,6,9);gupdate(2,0,6,6,9);gupdate(1,1,6,6,9);
gupdate(0,-1,6,6,9);
gupdate(-2,0,6,6,9);gupdate(0,1,6,6,9);gupdate(2,1,6,6,9);
```

A, B, -2, *A + , -2, *B

Order may be function of n.

array (1.. 1, [1])

array (1.. 18, [2, 2, 2, 2, so[5], so[6], so[7], so[8], so[9], so[10], so[11], so[12], so[13], so[14], so[15], so[16], so[17], so[18]])

A, B, 2, *A + , 0, *B

When a is even, order may be function of n.

When a is odd, order may be function of n.

When b is even, order may be function of n.

When b is odd, order may be function of n.

array (1 .. 7, [4, -8, 12, -12, 8, -4, 1])

array (1 .. 18, [2, 14, 22, 16, 14, 46, 106, 154, 206, 382, 784, 1382, 2222, 3794, 6962, 12526, so[17], so[18]])

array (1 .. 8, [1, 2, -2, -2, 2, 2, -1, -1])

array (1 .. 18, [2, 10, 2, 2, 2, 10, 16, 34, 38, 50, 46, 50, 54, 80, 122, 194, 274, 370])

array (1 .. 6, [3, -5, 7, -5, 3, -1])

array (1 .. 18, [2, 2, 2, 16, 38, 46, 54, 122, 274, 458, 688, 1198, 2302, 4142, so[15], so[16], so[17], so[18]])

array (1 .. 7, [0, 2, 2, -2, -2, 0, 1])

array (1 .. 18, [2, 10, 14, 2, 22, 10, 16, 34, 14, 50, 46, 50, 106, 80, 154, 194, so[17], so[18]])

A, B, 1, *A + , 1, *B

Order is independent of n and parities of a and b.

A, B, 0, *A + , -1, *B

When b is even, order may be function of n.

When b is odd, order may be function of n.

array (1 .. 2, [5, -4])

array (1 .. 18, [2, 14, 62, 254, 1022, 4094, so[7], so[8], so[9], so[10], so[11], so[12], so[13], so[14], so[15], so[16], so[17], so[18]])

array (1 .. 2, [3, -2])

array (1 .. 18, [6, 10, 18, 34, 66, 130, so[7], so[8], so[9], so[10], so[11], so[12], so[13], so[14], so[15], so[16], so[17], so[18]])

A, B, -2, *A + , 0, *B

Order may be function of n.

array (1 .. 4, [1, 2, -1, -1])

array (1 .. 18, [2, 10, 8, 18, 22, 40, 58, 98, 152, 250, so[11], so[12], so[13], so[14], so[15], so[16], so[17], so[18]])

A, B, 0, *A + , 1, *B

H(, 13, -5, -5,) has infinite order.

H(, 17, -7, -7,) has infinite order.

H(, 25, -11, -11,) has infinite order.

H(, 29, -13, -13,) has infinite order.

H(, 37, -17, -17,) has infinite order.

H(, 41, -19, -19,) has infinite order.

H(, 19, -5, -5,) has infinite order.

H(, 23, -7, -7,) has infinite order.

H(, 31, -11, -11,) has infinite order.

H(, 35, -13, -13,) has infinite order.

H(, 43, -17, -17,) has infinite order.

H(, 47, -19, -19,) has infinite order.

When b is even, order may be function of n.

When b is odd, order may be function of n.

array (1 .. 3, [0, 0, 1])

array (1 .. 18, [2, 8, 2, 2, 8, 2, 2, 8, so[9], so[10], so[11], so[12], so[13], so[14], so[15], so[16], so[17], so[18]])

array (1 .. 3, [2, -2, 1])

array (1 .. 18, [6, 2, 0, 2, 6, 8, 6, 2, so[9], so[10], so[11], so[12], so[13], so[14], so[15], so[16], so[17], so[18]])

A, B, 2, *A + , 1, *B

When b is even, order may be function of n.

When b is odd, order may be function of n.

array (1 .. 6, [-1, 3, 7, 3, -1, -1])

array (1 .. 18, [2, 2, 22, 16, 74, 134, 262, 682, 1226, 2966, 6064, 13066, 28622, 60014, so[15], so[16], so[17], so[18]]   )

array (1 .. 7, [2, -2, 4, -4, 2, -2, 1]   )

array (1 .. 18, [6, 2, 18, 18, 6, 26, 48, 34, 54, 122, 138, 162, 318, 464, 558, 898, so[17], so[18]]   )

The fact that recurrence relations exist for the H(a,b,b) groups with the parity of b fixed seems unlikely at first as these groups have order 2n, 6n, 8n and ∞ according to (a-b) mod 6. However it is easy to see that this result is correct :-

Consider 4 cases, remembering that (a,b)=1.

Case 1 : $b-a\equiv 0 \mod 6$ and the group is infinite. Here, clearly $a=6\alpha\pm 1$ and $b=6\beta\pm 1$ for some α and β.

Case 2 : $b-a\equiv 3 \mod 6$ and the order is 8n. Here a is not divisible by 3 and $b=6\lambda+3+a$.

Case 3 : $b-a\equiv 2,4 \mod 6$ and the order is 6n. Here a is not divisible by 2 and $b=6\lambda\pm 2+a$. Note here that b must therefore be odd.

Case 4 : $b-a\equiv 1,5 \mod 6$ and the order is 2n. Also $b=6\lambda\pm 1+a$.

If we know that b is odd then we can obtain the possible n values of all the cases.

For case 1 $n=a+2b=6\lambda\pm 3$ for some λ.

For case 2 a must be even as b is odd and $b=6\lambda+3+a$. So $n=12\lambda+6+3a$ or $n=6\mu$ for some μ.

For case 3 $n=12\lambda\pm 4+3a$ and a is odd so $n=6\mu\pm 1$.

For case 4 a must be even as b is odd so $n=12\lambda\pm 2+3a$ or $n=6\mu\pm 2$. This means that we have different cases for different n and as n increases from 1 the orders of the groups divided by n are {6,2,0,2,6,8,6,2,0,2,6,8,...} which satisfies the recurrence relation which the program found.

When b is even the result can be easily verified in the same way Here cases 1 and 3 are impossible as they require b to be odd, case 2 occurs when $n=6\mu+3$ and case 4 when $n=6\mu\pm2+3$ i.e. $n=6\mu\pm1$. As b is even here, a must always be odd and hence so must n. So the program calculates the order for even n to give the above results.

The results obtained for H(a,b,2a+b) are a rewriting of the results quoted in the known results section. Note that the latter consist of 4 different recurrence relations identified by a (mod 4). Now $n=3a+2b$ so if $a=4\alpha+i$ with $0\leq i\leq 3$ then $n=12\alpha+3i+2b$. Hence when b is even n-3i must be divisible by 4 so when $n\equiv0 \pmod 4$ i must be 0 , when $n\equiv1 \pmod 4$ i must be 3, when $n\equiv2 \pmod 4$ i must be 2 and when $n\equiv3 \pmod 4$ i must be 1. So the orders of H(a,b,2a+b) for even b are elements of 4 recurrence sequences taken consecutively and alternately. Hence the main sequence also satisfies a recurrence relation.

(f) <u>Some new results</u>

As mentioned in the general formula section, the derived group of H(a,b,c) is $K(a,b,c)= \langle X_1,\ldots X_n | Y_{i+a}=Y_iY_{i+a+b}, 1\leq i\leq 2n\rangle$, where $Y_i=X_i^{(n-(i \bmod (2n))+1/2)/|n-(i \bmod (2n))+1/2|}$, which is of index 2n in H(a,b,c). Let c=la+mb. Now consider

$$\begin{aligned} Y_{i+ta} &= Y_{i+(t-1)a}Y_{i+ta+b} \\ &= Y_{i+(t-1)a}Y_{i+(t-1)a+b}Y_{i+ta+2b} \end{aligned}$$

$$= Y_{i+(t-1)a} Y_{i+(t-1)a+b} Y_{i+(t-1)a+2b} Y_{i+ta+3b}$$

$$= Y_{i+(t-1)a} Y_{i+(t-1)a+b} \cdots Y_{i+(t-1)a+mb} Y_{i+ta+(m+1)b}. \qquad (1)$$

Now say

$$Y_{i+(t-1)a+(p-1)b} = Y_{i+(t-1)a}^{X(1,p-1)}\, Y_{i+(t-2)a}^{X(2,p-1)} \ldots Y_{i+(t-p)a}^{X(p,p-1)}. \qquad (2)$$

Therefore using this

$$Y_{i+(t-1)a+pb} = Y_{i+(t-1)a+(p-1)b} Y_{i+(t-2)a+(p-1)b}^{-1}$$

$$= Y_{i+(t-1)a}^{X(1,p-1)}\, Y_{i+(t-2)a}^{X(2,p-1)-X(1,p-1)} \ldots$$

$$Y_{i+(t-p)a}^{X(p,p-1)-X(p-1,p-1)}\, Y_{i+(t-p)-1a}^{-X(p,p-1)}.$$

So we see that $x(1,p)=x(1,p-1)$, $x(i,p)=x(i,p-1)-x(i-1,p-1)$ for $2 \le i \le p$ and $x(p+1,p)=-x(p,p-1)$, with $x(1,1)=1$ and $x(2,1)=-1$. So $x(i,j)=(-1)^{i+1} * {}^{j}C_{i-1.}$ This is easily proved by induction.

Applying the above to (1) we obtain :

$$Y_{i+ta} = Y_{i+(t-1)a}^{1+X(1,1)+X(1,2)\ldots+X(1,m)} Y_{i+(t-2)a}^{X(2,1)+X(2,2)+\ldots X(2,m)}$$
$$Y_{i+(t-3)a}^{X(3,2)+X(3,3)\ldots+X(3,m)} \ldots Y_{i+(t-m-1)a}^{X(m+1,m)} Y_{i+(t-L-1)a}^{-1}.$$

By induction it is seen that $x(i,i-1)+x(i,i)+\ldots+x(i,r)=-x(i+1,r+1)$. Fix i and assume this is true for $r \le k$ and. So we have that

$x(i+1,r+1)=x(i+1,r)-x(i,r)=-\{x(i,i-1)+x(i,i)+\ldots+x(i,r-1)\}-x(i.r)$. So for each i we have shown that the result holds by induction and so :

$$Y_{i+ta} = Y_{i+(t-1)a}^{-X(2,m+1)}\, Y_{i+(t-2)a}^{-X(3,m+1)}\, Y_{i+(t-3)a}^{-X(4,m+1)} \ldots$$

$$Y_{i+(t-m-1)a}^{-X(m+2,m+1)} Y_{i+(t-L-1)a}^{-1}. \qquad (3)$$

Without loss of generality take $L>m$ so that when the above terms are ordered with respect to the coefficient of a in their subscripts, the (t-L-1) term will appear last. Note that the above expression shows that Y_{i+ta} is expressible in terms of $Y_{i+La,}$ $Y_{i+(L-1)a}, \ldots, Y_i$ and let : $Y_{i+ta} = Y_{i+La}^{y(L,t)} Y_{i+(L-1)a}^{y(L-1,t)} \ldots Y_i^{y(0,t)}.$

Let $\varepsilon(i) = (-1)^i$. Now $Y_{1+ia+b} = Y_{1+ia+b+r((L+1)a+(m+1)b)}{}^{\varepsilon(r)} = Y_{1+a(i+r(L+1))+b(1+r(m+1))}{}^{\varepsilon(r)}$.

Assume the prime decomposition of m is $m=p(1)^{q(1)}...p(r)^{q(r)}$. We wish to solve the equation $\alpha L_1 a=(1-tm_1)b$ (*). Remember that $(a,b)=1$. This last fact implies that for all i $p(i)$ must not divide a. Let $L_1=p(1)^{u(1)}...p(r)^{u(r)}v$ where $(v,m_1)=1$. Then $p(1)^{u(1)}...p(r)^{u(r)} \mid b$. As $(va,m_1)=1$ there exist x and y such that $xva+ym_1=1$. Let $t=y$ and $\alpha=xb\ p(1)^{-u(1)}...p(r)^{-u(r)}$ then it is seen that these values solve (*) and so a solution exists iff $(a,m_1)=1$ and $(L_1/(b,L_1),m_1)=1$. In particular note that a solution exists in the special case $(L_1,m_1)=1$ and $(a,m_1)=1$.

In the cases in which a solution to (*) exists let $m+1= m_1$, $L+1= L_1$ and $r= \alpha-t$ and so the equation becomes $\alpha((L+1)a+(m+1)b) = b(1+r(m+1))$. So $Y_{1+a(i+r(L+1))+b(1+r(m+1))}{}^{\varepsilon(r)} = Y_{1+a(i+(\alpha-t)(L+1))}{}^{\varepsilon(t)}$.

Also
$$\begin{aligned}
Y_{1+a+b} &= Y_1{}^{-1}Y_{1+a},\\
Y_{1+2a+b} &= Y_{1+a}{}^{-1}Y_{1+2a}, \ldots,\\
Y_{1+La+b} &= Y_{1+(L-1)a}{}^{-1}Y_{1+La},\\
Y_{1+(L+1)a+b} &= Y_{1-mb}{}^{-1}\\
&= Y_{1-mb-a}{}^{-1}Y_{1-(m-1)b}{}^{-1}\\
&= Y_{1-mb-2a}{}^{-1}Y_{1-(m-1)b-a}{}^{-2}Y_{1-(m-2)b}{}^{-1}\\
&= \ldots\ldots\\
&= Y_{1-mb-ma}{}^{-1}Y_{1-(m-1)b-(m-1)a}{}^{-(m,1)}\ldots Y_{1-b-a}{}^{-(m,m-1)}Y_1{}^{-1}.
\end{aligned}$$

where $(i,j)={}^iC_j$.

Now $Y_{1-(m-j)b-(m-j)a}$

$=Y_{(1-mb)+jb-(m-j)a}$ (Use (2)).

$=Y_{1-mb-(m-j)a}{}^{(j,0)}Y_{1-mb-(m-(j-1))a}{}^{-(j,1)}Y_{1-mb-(m-(j-2))a}{}^{(j,2)}\ldots Y_{1-mb-ma}{}^{\varepsilon(j)(j,j)}$.

Therefore $Y_{1+(L+1)a+b}$

$=Y_{1-mb-ma}{}^{-1+(m,1)(1,1)-(m,2)(2,2)\ldots+\varepsilon(m)(m,m-1)(m-1,m-1)}$

$Y_{1-mb-(m-1)a}{}^{-(m,1)(1,0)+(m,2)(2,1)\ldots+\varepsilon(m+1)(m,m-1)(m-1,m-2)}$

$Y_{1-mb-(m-2)a}{}^{-(m,2)(2,0)+(m,3)(3,1)\ldots+\varepsilon(m)(m,m-1)(m-1,m-3)}$

$\ldots Y_{1-mb-a}{}^{-(m,m-1)(m-1,0)}Y_1{}^{-1}$.

Now $-(m,i)(i,0)+(m,i+1)(i+1,1)\ldots+(-1)^{m-i}(m,m-1)(m-1,m-1-i) = (-1)^{m-i}(m,i)=(-1)^{m-i}(m,m)(m,m-i)$. This is easily seen. (Consider

$$-b^{-m} = (1-(1+b)/b)^m$$

$$= {}^m\sum_{i=0} (-1)^i(m,i)(1+b)^i b^{-i}$$

$$= {}^m\sum_{i=0} (-1)^i(m,i)\ {}^i\sum_{r=0} b^{r-i}(i,r)$$

$$={}^m\sum_{j=0} b^{-j}\big((-1)^j(m,j)(j,0)+(-1)^{j+1}(m,j+1)(j+1,1)+\ldots$$

$+(-1)^m(m,m)(m,m-j)\big)$ and compare coefficients to obtain the result.) So we have that :

$Y_{1+(L+1)a+b}$

$=Y_{1-mb-ma}{}^{\varepsilon(m)(m,0)}Y_{1-mb-(m-1)a}{}^{\varepsilon(m-1)(m,1)}\ldots Y_{1-mb-a}{}^{-(m,m-1)}Y_1{}^{-1}$.

$=Y_{1+(L+1-m)a+b}{}^{\varepsilon(m+1)(m,0)}Y_{1+(L+2-m)a+b}{}^{\varepsilon(m)(m,1)}\ldots Y_{1+La+b}{}^{(m,m-1)}Y_1{}^{-1}$.

$=Y_{1+(L-m)a}{}^{\varepsilon(m)(m+1,0)}Y_{1+(L+1-m)a}{}^{\varepsilon(m-1)(m+1,1)}\ldots Y_{1+La}{}^{(m,m-1)}Y_1{}^{-1}$.

As the group is generated by $Y_{1+La}, Y_{1+(L-1)a},\ldots,Y_1$ (as gcd(a,b)=1, above we have seen that $Y_{1+(L+1)a+b}$ is expressible in terms of Y_{1+ta} terms using this argument inductively we can see that $Y_{1+ra+sb}$ is expressible in terms of $Y_{1+ra+(s-1)b}$ terms each of which are expressible in terms of Y_{1+ta} each of which are expressible in terms of the claimed generators. However for all i we see that

$Y_{1+i} = Y_{1+i(ax+by)} = Y_{1+(ix)a+(iy)b}$ which we have just shown to be expressible in terms of the claimed generators.) and it is also abelian we can write a relation matrix with the property that the determinant of this matrix is divisible by the order of the group. We now have 2 different expressions for Y_{1+ia+b} for i from 1 to L in terms of these generators as $Y_{1+ia+b} = {Y_{1+a(i+(\alpha-t)(L+1))}}^{\varepsilon(t)} = {Y_{1+La}}^{\varepsilon(t)y(L,i+(\alpha-t)(L+1))} {Y_{1+(L-1)a}}^{\varepsilon(t)\,y(L-1,i+(\alpha-t)(L+1))} \dots {Y_1}^{\varepsilon(t)\,y(0,i+(\alpha-t)(L+1))}$.

Let us now consider the y(i,j) numbers. $Y_{i+(t+1)a} = {Y_{i+La}}^{y(L,t+1)} {Y_{i+(L-1)a}}^{y(L-1,t+1)} \dots {Y_i}^{y(0,t+1)}$ by definition of the y(i,j). But $Y_{i+(t+1)a} = {Y_{i+ta}}^{(m+1,1)} {Y_{i+(t-1)a}}^{-(m+1,2)} {Y_{i+(t-2)a}}^{(m+1,3)} \dots {Y_{i+(t-m)a}}^{\varepsilon(m)(m+1,m+1)} {Y_{i+(t-L)a}}^{-1}$, from (3). We can now apply the defining equation of the y(i,j) to this and compare it with the above to obtain : $y(k,t+1) = (m+1,1)y(k,t) -(m+1,2)y(k,t-1)+(m+1,3)y(k,t-2)\dots$

$+(-1)^m(m+1,m+1)y(k,t-m)-y(k,t-L)$.

The initial values of these recurrence relations can be seen by looking at the defining equation with values of t from 0 to L : y(h,k)=1 if h=k or 0 otherwise, for h and k from 0 to L. Using the above information we can write down these y(i,j) in terms of one recurrence relation.

If y(i,i+k)=0 for k from 1 to L then the initial values of the recurrence sequence y(i,i+k) as k increases from 0 are the same as the initial values of the sequence y(0,k) and so y(i,i+k)=y(0,k) for all k. Assume i<L then y(i,i+k)=0 for k from 1 to L-i. Now if this is true for k from L-i to e<L then y(i,i+e+1)=0 implies that the coefficient of y(k,t-e) in the recurrence expansion is zero, i.e. m<e<L. So if the property holds it is required that m<e<L for all e from L-i to L-1 i.e. that i<L-m. So for i<L-m, y(i,i+k)=y(0,k).

Now we will take $i \geq L-m$. Here $y(i,L+1)=(-1)^{L-i}(1+m,1+L-i)$.

The sequence $y(i-1,t-1)+(-1)^{L-i+1}(m+1,L-i+1)y(0,t)$ is defined for $t \geq 1$, satisfies the same recurrence formula as $y(i,t)$ and for t from 1 to L+1 these two sequences have the same elements : so they are the same sequences. It is easy to use this recursively to find a general formula for when $L > i \geq L-m$.

$y(r,t) = (-1)^{L+1-r}(m+1,L+1-r)y(0,t)+\dots +(-1)^{m+1}(m+1,m+1)y(0,t+L-m-r)$
$+ \; y(L-m-1,t+L-m-r-1)$
$=(-1)^{L+1-r}(m+1,L+1-r)y(0,t)+\dots+(-1)^{m+1}(m+1,m+1)y(0,t+L-m-r)$
$+y(0,t-r)$.

Now $y(0,L+1)=-1$ and clearly $y(0,j)=0$ for j from 1 to L so $y(L,j) = -y(0,j+1)$. Call $y(0,j)=y(j)$.

Now we showed above that the order of $H(a,b,c)$ divided $2n|\det(A-B)|$ where

$$A=(-1)^t \begin{bmatrix} y(L,1+(\alpha-t)(L+1)) & \dots & y(0,1+(\alpha-t)(L+1)) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ y(L,L+1+(\alpha-t)(L+1)) & \dots & y(0,L+1+(\alpha-t)(L+1)) \end{bmatrix}$$

and

$$B=\begin{bmatrix} 0 \cdots & & & & & 0 & 1 & -1 \\ 0 \cdots & & & & & 1 & -1 & 0 \\ & & & & \cdot & \cdot & & \\ & & & \cdot & \cdot & & & \\ & & \cdot & \cdot & & & & \\ 1 & -1 & 0 \cdots & & & & & 0 \\ (m,m-1) & -(m+1,m-1) & (m+1,m-2) & \cdots & (-1)^m(m+1,0) & 0 \cdots & 0 & -1 \end{bmatrix}$$

Rewrite A in terms of y(j) . Let $y((\alpha-t)(L+1)+i)=g(i)$. While $c \leq m-1$ add $(-1)^{L-r+c}(m+1,L-r+1+c)$ times the cth to all the columns r from L-m+c to L-1 starting with c=0 and continuing until the condition ceases to hold. Then multiply the (L+1)st column (i.e. the one on the far left) by -1 and move it to the far right.

This produces new matrices for A and B :

$$A=(-1)^t \begin{bmatrix} g(2-L) & g(3-L) & \cdots & g(2) \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ g(2) & g(3) & \cdots & g(2+L) \end{bmatrix}$$

$$\text{Let } C=\begin{bmatrix} 0 \cdots & & & & & 0 & 1 & -1 \\ 0 \cdots & & & & & 1 & -1 & 0 \\ & & & & \cdot & \cdot & & \\ & & & \cdot & \cdot & & & \\ & & \cdot & \cdot & & & & \\ 1 & -1 & 0 \cdots & & & & & 0 \\ m & 0 \cdots & & & & & 0 & -1 \end{bmatrix}$$

To find B note that adding the columns for c from 0 to r<L-m

produces the matrix C+D where D has r+1 none zero rows and :

$D[s,i]=(-1)^{i+s-1}(m+2,i+s)$ for i from 2 to m+2-s and s from 1 to r .

$D[r+1,i]=(-1)^{i+r}(m+1,i+r)$ for i from 2 to m+1-r .

$D(i,j)=0$ otherwise.

(Call the above matrix $D_1(r)$.)

If we add the columns for c from 0 to $2L-2m>r\geq L-m$ then we would obtain $C+D_1(r)+D_2(r)$ where the latter has zero terms except for :

$$D_2(r)[s,i]=(-1)^{L+i+s-1}((m+1,L+i+s-1-m)^*(m+2,m+2)+\ldots$$
$$+(m+1,L+i-m+u)^*(m+2,m+s+1-u)) \qquad (*)$$

where $u=\min\{2m-L+1-i,r-(L-m)\}$, i is from 2 to 2m-L+2-s and s is from 1 to 1+r-(L-m). Here we can use the fact that (c+d,e) = (c,0)(d,e) + (c,1)(d,e-1) + ...(c,e)(d,0) (compare coefficients of the binomial expansion of $(1+a)^c$ times the expansion of $(1+a)^d$ with the expansion of $(1+a)^{c+d}$). Now if (*) is of this form then as $L+i-m+u\geq L+i+s-1-m$ (the minimum value of 2m-L+1-i is s-1) we must subtract the second binomial parameters from m+1. So we must have that (m+1)-(L+i+s-1-m)=e, (m+1)-(L+i-m+u)=0 and (m+2)-(m+s+1-u)=e for some e. Clearly the last equation is redundent, so the requirement is that u=2m-L-i+1 i.e. that $2m-L-i+1\leq r-(L-m)$ or $r+i\geq m+1$. (Note that when r=m-1 then this condition is always satisfied). So in these cases we see that

$$\begin{aligned} D_2(r)[s,i] &=(-1)^{L+i+s-1}(2m+3,u+1-s) \\ &=(-1)^{L+i+s-1}(2m+3,-s+2m-L-i+2) \\ &=(-1)^{L+i+s+1}(2m+3,s+L+i+1). \qquad (**) \end{aligned}$$

If we then need to add up to the rth column where $3L-3m>r\geq 2L-2m$ then we would have $C+D_1(r)+D_2(r)+D_3(r)$. We can find

$D_3(r)$ using an inductive argument : if we already know $D_3(r-1)$ then we need to add the rth column of the matrix to the columns from L-m+r to L-1. So we need to know what the rth column of the $D_2(r-1)$ matrix is. Now the rth column of this matrix has column index $i=(L+1)-r$ and we have added columns up to r-1 so `r+i`=(r-1)+(L+1-r)=L which is always bigger than or equal to m+1. So all the terms in the rth column of $D_2(r-1)$ are expressible in the simple form (**). This enables us to write down the matrix $D_3(r)$.
We wish to identify the cth column of $D_2(c-1)$. $D_2(c-1)[s,i] =(-1)^{L+i+s+1}(2m+3,s+L+i+1)$. Now the cth column has column index $i=L+1-c$ and so we can substitute this in the expression of $D_2(c-1)[s,i]$. We know that the non-zero terms in this matrix have i from 2 to 2m-L+2-s so $i \leq 2m-L+2-s$ and hence $s \leq 2m-L+2-i = 2m-L+2-(L+1-c) = 2m-2L+1+c$. So therefore we need to add $(-1)^{L-r+c}(m+1,L-r+1+c)*(-1)^{2L+2+s-c}(2m+3,2L+2+s-c)$ to the sth row of the rth column where s is from 1 to 2m-2L+1+c and r is from L-m+c to L-1. This is $(-1)^{3L+2+s-r}(m+1,L-r+1+c)(2m+3,2L+2+s-c)$. Again noting that $i=L+1-r$, the bounds for s and i (r) just quoted give the columns that contribute to the [s,i] term : $m+1-i \geq c \geq s+2L-2m-1$, also $c \leq R$.

$$D_3(R)[s,i]=(-1)^{2L+1+s+i}((m+1,i+s+2L-2m-1)*(2m+3,2m+3)+\dots$$
$$+(m+1,L+i-m+u)*(2m+3,m+L+s+2-u))$$

where $u=\min\{2m-L+1-i,R-(L-m)\}$, i is from 2 to 3m-2L+2-s and s is from 1 to 1+R-(2L-2m) ; the other entries in the matrix are zero.

Arguing as before we see that when $R+i \geq m+1$, we have a simple expression for the term, this time with $e=m-L+1-s+u$

$$D_3(R)[s,i] =(-1)^{2L+i+s+1}(3m+4,-s+3m-2L-i+2)$$

$=(-1)^{2L+i+s+1}(3m+4,s+2L+i+2)$.

We can generalise this result to obtain :

$D_a(r)[s,i]=$

$(-1)^{(a-1)L+i+s+1}((m+1,(a-1)(L-m)+i+s-1)^*((a-1)m+a,(a-1)m+a)+\ldots$

$+(m+1,L+i-m+u)^*((a-1)m+a,m+s+1-u+(a-2)(L+1))$ where $u=\min\{2m-L+1-i,r-(L-m)\}$, i is from 2 to $am-(a-1)L+2-s$ and s is from 1 to $1+r-(a-1)(L-m)$, and all other terms are zero. Again we obtain the simple form when $r+i\geq m+1$:

$D_a(r)[s,i]=(-1)^{(a-1)L+i+s+1}(am+a+1,s+(a-1)L+i+(a-1))$.

As we mentioned above, the simple case applies to all entries in all the matrices when $r=m-1$. So we now know the form of the matrix B after we have added the required multiples of the columns 0 to m-1. The next step is to multiply the left hand column by -1 and move it to the right .This enables us to write down the following theorem :

Theorem 2

The order of H(a,b,c) divides $2n|\det(A-B)|$ where A is as above and B as below.

$$\text{Let } C_0=\begin{bmatrix} 0 \;\ldots\; & & & 0 & 1 & -1 & 0 \\ 0 \;\ldots\; & & & 1 & -1 & 0 & 0 \\ & & & \cdot & \cdot & & \\ & & \cdot & \cdot & & & \\ & \cdot & \cdot & & & & \\ 1 & -1 & & & & \ldots & 0 \\ -1 & 0 & & & & 0 & -1 \\ 0\ldots & & & & 0 & -1 & -m \end{bmatrix}$$

Let b be the least integer satisfying $m-1<b(L-m)$. Let C_f have zero terms unless i is from 1 to $fm-(f-1)L+1-s$ and s is from 1 to

$m-(f-1)(L-m)$, when $C_f[s,i] = (-1)^{(f-1)L+i+s}(fm+f+1, s+(f-1)L+i+f)$.

Let $B=C_0+C_1+\ldots+C_b$.

For example :- The group $H(-1,3,2)$.

This group is in the class $c=4a+2b$ with $n=4$. To calculate $2n|\det(A-B)|$ we need to know L, m, n, α and t.

Now α and t satisfy $\alpha(L+1)a=(1-t(m+1))b$. Substituting the known values we obtain $-5\alpha=3-t9$. As $3|\alpha$ let $\alpha=3q$ which gives us $-5q=1-3t$. This is solved by $q=1$, $t=2$ so to get a complete solution set we let $q=1+j$, $t=2+i$. It is easy to see that $i=5å$, $j=3å$ for some integer $å$. This gives possible values of $\alpha-t=1+4å$.

Thus $g(i) = y(5+20å+i)$ and $y(t+1) = 3y(t)-3y(t-1)+y(t-2)-y(t-4)$. Note that $(-1)^t=(-1)^{å}$. Take $å=0$, so $g(i)=y(5+i)$ and we require the values of $y(5+i)$ for i from -2 to 6. We know that $y(0)=1$ and $y(1)=y(2)=y(3)=y(4)=0$. So the first few members of the sequence are $\{1,0,0,0,0,-1,-3,-6,-10,-15,-20,-22\}$.

So $A=$

$$\begin{bmatrix} 0 & 0 & -1 & -3 & -6 \\ 0 & -1 & -3 & -6 & -10 \\ -1 & -3 & -6 & -10 & -15 \\ -3 & -6 & -10 & -15 & -20 \\ -6 & -10 & -15 & -20 & -22 \end{bmatrix}$$

To find B we note that $L-m=2>1=m-1$ and so we need just C_1. Now this is $C_1[s,i]=(-1)^{i+s}(4,s+i+1)$ where i is from 1 to $3-s$ and s is from 1 to 2.

So B is

$$\begin{bmatrix} 4 & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}.$$

Also t is even, so we look at the determinant of A-B : which is 41*9. The order of the group, 72 divides this multiplied by 2n.

In particular when m=0 the order is often obtainable. In this case y(i,i+k)=y(0,k) for i<L and y(L,j)=-y(0,j+1). So only the rearrangement step has to be carried out. Note that here the recurrence relation is very simple also : y(0,t)=y(0,t-1)-y(0,t-L-1).

Example :- Take L=3,m=0,a=1 and b=3 then n=7.

The first few elements of the recurrence relation sequence are {1,0,0,0,-1,-1,-1,-1,0,1,2,3,3,2,0,-3,-6,-8,...}. α-t=6+7å : take å=0. So the top left hand corner entry in A is y(0,21).

Hence A is :

$$\begin{bmatrix} 9 & 17 & 22 & 21 \\ 17 & 22 & 21 & 12 \\ 22 & 21 & 12 & -5 \\ 21 & 12 & -5 & -27 \end{bmatrix}$$

B is :

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$ and t is odd.

The determinant of this matrix is 6399 and 2n times this is divisible by the order of the group. If we chose å=-1 then t is still odd but α-t=-1, so the top left hand entry in the matrix is y(0,-7). Here A is :

$$\begin{bmatrix} -2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

Here 2n|det(A-B)|=42 the order of the group.

In fact a simple formula for the cases when m=L-1 can also be obtained.

Consider the groups in the classes c=La+(L-1)b. The formula for these groups is complicated because m is big (m=L). Note however that $K(a,b,c) = \langle X_1,\ldots X_n \mid Y_{i+a}=Y_iY_{i+a+b},\ 1 \le i \le 2n \rangle$ is $J(a`,b`,c`) = \langle X_1,\ldots, X_n \mid Y_{i+a`}Y_i=Y_{i+a`+b`},\ 1\le i\le 2n\rangle$ where a`=a+b and b`=-b. Now if n=(L+1)a+Lb then n=(L+1)a`+b`. It is possible to follow through a very similar argument to that in Theorem 2 to obtain a formula for J(a,b,c) and the groups c=La+(L-1)b are the c`=La` groups of this new type and the formula for the c`=La` groups will be simple.

Proposition 3

H(a,b,La+mb)≡H(a,b,-(L+2)a-(m+2)b)

Proof :-

H(a,b,La+mb) has K(a,b,La+mb)

$= \langle X_1,\ldots, X_{(L+1)a+(m+1)b} \mid Y_{i+a}=Y_iY_{i+a+b}\ i:=1,\ldots,2((L+1)a+(m+1)b) \rangle$ as a subgroup of index 2n. Note that the latter is

$= \langle X_1,\ldots, X_{(L+1)a+(m+1)b} \mid Y_{j-b}=Y_{j-a-b}Y_j\ i:=1,\ldots,2((L+1)a+(m+1)b) \rangle$

= K(-b,-a,(L+1)a+(m+1)b-(-a-b)) which is of index 2n in

H(-b,-a,(L+2)a+(m+2)b) ≡ H(a,b,-(L+2)a-(m+2)b). It is easy to construct an isomorphism between these groups. Take an element t of H(a,b,La+mb). Say that this element is in the coset of K(a,b,La+mb) which is represented in the Schreier transversal by i. Map the element to the same member of K(-b,-a,(L+2)a+(m+2)b) as ti^{-1}. Then map this to itself multiplied by the coset i.

$|H(a,b,La+mb)|=2n\ |K(a,b,La+mb)|$

$=2n\ |\ \langle X_1,\ldots,X_{(L+1)a+(m+1)b}|\ Y_{i+a}=Y_iY_{i+a+b}\ \ i:=1,\ldots,2((L+1)a+(m+1)b)\ \rangle\ |$

$=2n\ |\ \langle X_1,\ldots,X_{(L+1)a+(m+1)b}|\ Y_{j-b}=Y_{j-a-b}Y_j\ \ i:=1,\ldots,2((L+1)a+(m+1)b)\ \rangle\ |$

$=2n\ \ |K(-b,-a,(L+1)a+(m+1)b-(-a-b))|$

$=2n\ \ |H(-b,-a,(L+2)a+(m+2)b)|=2n\ \ |H(a,b,-(L+2)a-(m+2)b)|.$

(g) Conjecture

(1) The orders of the groups H(a,b,La+(L-1)b) with the parities of a and b fixed depend only on n=(L+1)a+Lb.

(2) The orders of the groups H(a\`,b\`,La\`) with the parities of a\` and b\` fixed are also a function of n\`=(L+1)a\`+b\`.

(3) When L is odd, a is of the same parity as a\`, b is of the same parity as b\` and n=n\` then the groups H(a,b,La+(L-1)b) and H(a\`,b\`,La\`) have the same order.

(4) The above two groups are in fact isomorphic.

To prove (4) it would be sufficient to show that the K subgroups of these two groups are isomorphic (as then it would be easy to construct an isomorphism between the H groups). Note as above that K(a,b,La+(L-1)b)≡J(a+b,-b,L(a+b)). So we would need to show that K(a\`,b\`,La\`)≡J(a+b,-b,L(a+b)) where 2|(a-a\`), 2|(b-b\`) and

(L+1)a+Lb=(L+1)a\`+b\`. This is :

$\langle X_1,\ldots X_{(L+1)a`+b`} \mid Y_{i+a`}=Y_iY_{i+a`+b`}, 1\leq i\leq 2((L+1)a`+b`)\rangle$ is $\langle X_1,\ldots X_{(L+1)a``+b``} \mid Y_{i+a``}Y_i=Y_{i+a``+b``}, 1\leq i\leq 2((L+1)a``+b``)\rangle$ where a``=a+b and b``=-b, a`` and a` are of the same parity, as are b`` and b` and (L+1)a`+b=(L+1)a``+b``. If we had already proved (1) and (2) then we could identify a` and a``, and b` and b`` ; hence it would be necessary to prove that J(a,b,La)≡K(a,b,La) for odd L.

REFERENCES

[1] C.M.Campbell, *Topics in the theory of groups*, (Lecture notes University of St Andrews, 1988).

[2] C.M.Campbell, H.S.M.Coxeter and E.F.Robertson, Some families of finite groups having two generators and two relations, *Proc. Roy. Soc. London A.* **357**(1977), 423-438.

[3] C.M.Campbell and E.F.Robertson, Deficiency zero groups involving Fibonacci and Lucas numbers, *Proc. Roy. Soc. Edinburgh Sect. A.* **81**(1978), 273-286.

[4] C.M.Campbell and R.M.Thomas, On (2,n)-Groups related to Fibonacci Groups, *Israel J. Math.* **58** No.**3** (1987), 370-380.

[5] R.D.Carmichael, Abstract definitions of the Symmetric and Alternating groups and certain other permutation groups, *Quart. J. Maths.* **49**(1923),226-270.

[6] D.L.Johnson, *Presentations of groups*, (London Math. Soc. Lecture Note Series **22**, 1976).

[7] D.L.Johnson, *Topics in the Theory of Group Presentations* (London Math. Soc. Lecture Note Series **42**, 1980).

[8] J.Neubuser, An elementary Introduction to coset table methods in computational group theory, *Groups-St Andrews* 1981, (London Math. Soc. Lecture Note Series **71**,1981).

[9] E.F.Robertson, *Symbolic Computation*, (Lecture notes University of St Andrews,1988).

[10] E.F.Robertson and C.M.Campbell, Symmetric Presentations, *Proc. 1987 Singapore Conference,* 497-506.

[11] R.M.Thomas, Some infinite Fibonacci Groups, *Bull. London Math. Soc.* **15**(1983), 384-386.